MACROSCOPIC ENERGY DIFFUSION FOR A CHAIN OF ANHARMONIC OSCILLATORS

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Abstract. We study the energy diffusion in a chain of anharmonic oscillators where the Hamiltonian dynamics is perturbed by a local energy conserving noise. We prove that under diffusive rescaling of space-time, energy fluctuations diffuse and evolve following an infinite dimensional linear stochastic differential equation driven by the linearized heat equation. We also give variational expressions for the thermal diffusivity and some upper and lower bounds.

1. Introduction

The deduction of the heat equation or the Fourier law for the macroscopic evolution of the energy through a diffusive space-time scaling limit from a microscopic dynamics given by Hamilton or Schrödinger equations, is one of the most important problem in non-equilibrium statistical mechanics ([5]). One dimensional chains of oscillators have been used as simple models for this study. In the context of the classical (Hamiltonian) dynamics, it is clear that non-linear interactions are crucial for the diffusive behavior of the energy. In fact, in a chain of harmonic oscillators the energy evolution is ballistic ([17]). In this linear system, the energy of each mode of vibration is conserved. Non-linearities introduce interactions between different modes and destroy these conservation laws and give a certain ergodicity to the microscopic dynamics.

In order to describe the mathematical problem, let us introduce some notation we will use in the rest of the paper. We study a system of anharmonic oscillators, each is denoted by an integer i. We denote by (q_i, p_i) the corresponding position and momentum (we set the mass equal to 1). Each pair of consecutive particles $(i, i+1)$ are connected by a spring which can be anharmonic. The interaction is described by a potential energy $\bar{V}(q_{i+1} - q_i)$. We assume that $\bar{V}$ is a nonnegative smooth function satisfying

$$Z_\beta := \int_{\mathbb{R}} e^{-\beta \bar{V}(r)} dr < \infty$$

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for all $\beta > 0$. Let a be the equilibrium inter-particle distance, where $\bar{V}$ attains its minimum that we assume to be $0 : \bar{V}(a) = 0$. It is convenient to work with inter-particle distances as coordinates, rather than absolute particle positions, so we define $r_j = q_j - q_{j-1} - a$. We denote the translated function $\bar{V}(\cdot + a)$ by $V(\cdot)$ hereafter. Namely, we assume $V(0) = 0$. The configuration of the system is given by $\{p_j, r_j\}_j$, and energy function (Hamiltonian) defined for each configuration is formally given by

$$H = \sum_j \mathcal{E}_j, \qquad \mathcal{E}_j = \frac{1}{2}p_j^2 + V(r_j).$$

The choice of $\mathcal{E}_j$ as the energy of each oscillator is a bit arbitrary, because we associate the potential energy of the bond $V(r_j)$ to the particle j. Different choices can be made, but this one is notationally convenient.

The corresponding Hamiltonian dynamics is given by the equations of motion:

$$\begin{aligned} r_j'(t) &= p_j(t) - p_{j-1}(t), \\ p_j'(t) &= V'(r_{j+1}(t)) - V'(r_j(t)). \end{aligned} \tag{1.1}$$

We are interested in the macroscopic evolution of the empirical energy profile under a diffusive macroscopic space-time scaling. More precisely the limit, as $N \to \infty$, of the energy distribution on $\mathbb{R}$ defined by

$$\frac{1}{N}\sum_i \mathcal{E}_i(N^2 t)\delta_{i/N}(dy). \tag{1.2}$$

Energy is not the only conserved quantity under the dynamics (1.1). Formally length and momentum are also integral of the motion. In one dimensional system, even for anharmonic interaction, generically we expect a superdiffusion of the energy, essentially because of the momentum conservation ([16, 1]). Adding a pinning potential $U(q_i)$ on each particle, it will break the translation invariance of the system and the momentum conservation, and we expect a diffusive behavior for the energy, i.e. the energy profile defined by (1.2) would converge to the solution $e(t,y)$ of a heat equation

$$\partial_t e = \partial_y \left(D(e)\partial_y e\right) \tag{1.3}$$

under specific conditions on the initial configuration. The diffusivity $D = D(e)$ is defined by the Green-Kubo formula associated to the corresponding infinite dynamics in equilibrium at average energy e (see below for the definition).

As the deterministic problem is out of reach mathematically, it has been proposed an approach that models the chaotic effects of the non-linearities by stochastic perturbations of the dynamics that conserves energy. In the harmonic case, random exchanges of momentum of nearest neighbor particles that conserve total energy but not momentum have been studied ([4, 8, 3]). Stochastic exchanges that also conserve total momentum have been considered in [1, 2], where a divergence of the diffusivity is proven for unpinned harmonic chains. The stochastic perturbations considered in

these papers are very degenerate (of hypoelliptic type), since they act only on the momenta of the particles, and not on the positions. In particular these stochastic dynamics conserve also the total length $\sum_j r_j$.

In this article we want to deal with anharmonic chains with noise that conserves only energy. For reasons we will explain in a moment, we need more elliptic stochastic perturbations that act also on the positions. In the case of one-dimensional unpinned chains, there is a way to define these perturbations locally (see the next section for details) just using squares of vector fields that appear in the Liouville vector field that generates the Hamiltonian dynamics. With these perturbations, we have a dynamics that conserves *only* the total energy. As a result, the dynamics has a one-parameter family of invariant measures given by the grand canonical measures $\{\nu_\beta, \beta > 0\}$ defined by

$$\nu_\beta = \prod_j \frac{e^{-\beta \mathcal{E}_i}}{\sqrt{2\pi\beta^{-1}} Z_\beta} dp_j dr_j.$$

Notice that $\{r_j, p_j\}_j$ are independently distributed under these probability measures.

So we can consider the system starting with the equilibrium distribution ν_β at temperature $T = \beta^{-1}$. We can prove the diffusive scaling limit by the results in this article in the following linearized sense: define the space time energy covariance in equilibrium at temperature β^{-1}:

$$C(i,j,t) = \mathbb{E}\left(\mathcal{E}_i(t)\mathcal{E}_j(0)\right) - e(\beta)^2$$

where $\mathbb{E}$ denotes the expectation for the stochastic dynamics starting with the grand-canonical measure at temperature β^{-1} and $e(\beta)$ is the expectation value of $\mathcal{E}_0$ under ν_β. In the following we will denote simply by e the corresponding value. Clearly $C(i,j,0) = \delta_{i,j}\chi(\beta)$, where $\chi(\beta)$ is the variance of $\mathcal{E}_0$ under ν_β. Then it follows by our results that

$$\lim_{N\to\infty} NC([Ny],[Nx],N^2t) \;=\; \frac{1}{\sqrt{2\pi Dt}} e^{-(x-y)^2/2\mathcal{D}t}$$

weakly, in the sense of the convergence of $N^{-1}\sum_{i,j} G(i/N)F(j/N)C(i,j,N^2t)$ for good test functions G and F of $\mathbb{R}$. Here $D = D(e)$ which is formally given by

$$D = \lim_{t\to\infty} \frac{1}{t} \sum_{i\in\mathbb{Z}} i^2 C(i,0,t). \tag{1.4}$$

We actually prove a stronger result at the level of fluctuation fields. If the system is in equilibrium at temperature β^{-1}, then standard central limit theorem for independent variables tells us that as $N\to\infty$ energy has Gaussian fluctuations, i.e. the energy fluctuation field

$$Y^N = \frac{1}{\sqrt{N}} \sum_i \delta_{i/N}\left\{\mathcal{E}_i(0) - e\right\}$$

converges in law to a delta correlated centered Gaussian field Y

$$\mathbb{E}[Y(F)Y(G)] = \chi \int F(y)G(y)dy.$$

In this article we prove that these *macroscopic* energy fluctuations evolve diffusively in time (after a diffusive space-time scaling) , i.e. that the time dependent distribution

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_i \delta_{i/N} \left\{ \mathcal{E}_i(N^2 t) - e \right\}$$

converges in law to the solution of the linear SPDE

$$\partial_t Y = D\, \partial_y^2 Y\ dt + \sqrt{2D\chi}\ \partial_y B(y,t)$$

where B is the standard normalized space-time white noise. In this sense, in equilibrium, energy fluctuation evolves macroscopically following the linearized heat equation.

The main point in the proof of this result is the following. Since total energy is conserved, locally the energy of each particle is changed by the energy currents with its neighbors, i.e. applying the generator L of the process to the energy $\mathcal{E}_i$ we obtain

$$L\mathcal{E}_i = W_{i-1,i} - W_{i,i+1} \tag{1.5}$$

where $W_{i,i+1} = -p_i V'(r_{i+1}) + W^S_{i,i+1}$. Here $-p_i V'(r_{i+1})$ is the instantaneous energy current associated to the Hamiltonian mechanism, while $W^S_{i,i+1}$ is the instantaneous energy current due to the stochastic part of the dynamics. While (1.5) provides automatically one space derivative already at the microscopic level, $W_{i,i+1}$ is not a space-gradient. In this sense this model falls in the class of the *non-gradient models*. Some of these non-gradient models have been studied with a method introduced by Varadhan [19]. The main point of this method is to prove that $W_{i,i+1}$ can be approximated by a fluctuation-dissipation decomposition

$$W_{i,i+1} \sim D\nabla\mathcal{E}_i + LF$$

for a properly chosen sequence of local functions F. In the harmonic case of our model, this decomposition is exact for every configuration, i.e. there exists a local second order polynomial F such that $W_{i,i+1} = D\nabla\mathcal{E}_i + LF$ for a constant D (cf. Remark 11.1 that contains an equivalent decomposition). In the anharmonic case, such decomposition can be only approximated by a sequence of local functions F_K in the sense that the difference has a small space-time variance with respect to the dynamics in equilibrium at given temperature (consequently D is a function of this temperature).

In order to do such decomposition, we have to use Varadhan's approach to non-gradient systems [19] and the generalization to non-reversible dynamics [20, 13]. The main ingredients of the methods are a *spectral gap* for the stochastic part of the dynamics, and a *sector condition* for the generator L of the dynamics. It is in order to prove these properties that we need such elliptic noise acting also on the positions.

We have to limit ourselves to these results on the equilibrium fluctuation and we are not able to prove the full non-linear equation (1.3) starting from a global non-stationary profile. Unfortunately most of known techniques to prove such hydrodynamic limits in diffusive scaling are based on relative entropy techniques (cf. [10], [21], [12]) that do not work for the energy diffusion in this model.

The article is organized as follows: In Section 2 we introduce our model and state main results. In Section 3, we give the strategy for proving the convergence of the finite dimensional distribution. The complete proof is divided into several sections 4, 5, 6, 9 and 13, with sector condition proved in Section 8, and the spectral gap in Section 12. The tightness shown in Section 7 concludes the proof. Variational expressions for the thermal diffusivity are obtained in Section 10 and some bounds on it are proven in Section 11.

2. Model and results

We will now give a precise description of the dynamics. We consider a system of N anharmonic oscillators in one-dimensional space, whose hamiltonian dynamics is perturbed by a random dynamics that conserves total energy. We consider a periodic boundary condition, but the results can be generalized to different boundary conditions or also to the infinite system.

Let $\mathbb{T} := (0,1]$ be the 1-dimensional torus, and for a positive integer N denote by $\mathbb{T}_N$ the lattice torus of length N : $\mathbb{T}_N = \{1,\dots N\}$. The configuration space is denoted by $\Omega^N = (\mathbb{R}^2)^{\mathbb{T}_N}$ and a typical configuration is denoted by $\omega = (p_i, r_i)_{i\in\mathbb{T}_N}$ where $r_i = q_i - q_{i-1}$ represents the inter-particle distance between the particles $i-1$ and i (here we assume $a = 0$ without loss of generality), and p_i represents the velocity of the particle i. All particles have mass equal to 1. The configuration changes with time and, as a function of time evolves as a Markov process in $\mathbb{R}^{2N}$ with infinitesimal generator given by

$$L_N^\gamma = A_N + \gamma S_N$$

where

$$A_N = \sum_{i\in\mathbb{T}_N}(X_i - Y_{i,i+1}), \quad S_N = \frac{1}{2}\sum_{i\in\mathbb{T}_N}\{(X_i)^2 + (Y_{i,i+1})^2\},$$

$$Y_{i,j} = p_i\partial_{r_j} - V'(r_j)\partial_{p_i}, \qquad X_i = Y_{i,i},$$

and $N+1 \equiv 1$. Notice that A_N is the generator of the Hamiltonian dynamics (the Liouville operator) while S_N is the generator of the stochastic perturbation. Here $\gamma > 0$ is the strength of the stochastic perturbation. We do not need any condition on γ, as long as it is strictly positive.

We assume that the function $V : \mathbb{R} \to \mathbb{R}_+$ satisfies the following three properties:

(i) $V(r)$ is a smooth symmetric function.
(ii) $0 < \delta_- \le V''(r) \le \delta_+ < +\infty$.
(iii) $\delta_-/\delta_+ > (3/4)^{1/16}$.

Remark 2.1. *The assumption (iii) is quite technical and required only in the proof of the spectral gap estimate in Section 12.*

We denote the energy associated to the particle i by

$$\mathcal{E}_i = \frac{p_i^2}{2} + V(r_i)$$

and the total energy defined by $\mathcal{E} = \sum_{i\in\mathbb{T}_N} \mathcal{E}_i$ which denotes the Hamiltonian of the original deterministic dynamics.

Remark 2.2. *The total energy satisfies $L_N^\gamma(\mathcal{E}) = 0$, i.e. total energy is a conserved quantity.*

Remark 2.3. *The other important conservation laws of the Hamiltonian dynamics, for the total length $\sum_{i\in\mathbb{T}_N} r_i$ and the total momentum $\sum_{i\in\mathbb{T}_N} p_i$, are destroyed by the stochastic noise S_N. In fact $L_N(\sum_i r_i) = \gamma S_N(\sum_i r_i) = -\gamma \sum_i V'(r_i)$, and $L_N(\sum_i p_i) = \gamma S_N(\sum_i p_i) = -\frac{\gamma}{2}\sum_i (p_{i-1}+p_i)V''(r_i)$.*

We define a probability measure ν_β^N on Ω^N by

$$\nu_\beta^N(dpdr) = \prod_{i=1}^N \frac{\exp\left(-\beta\left(\frac{p_i^2}{2} + V(r_i)\right)\right)}{\sqrt{2\pi\beta^{-1}}Z_\beta} dp_i dr_i$$

where

$$Z_\beta := \int_{\mathbb{R}} e^{-\beta V(r)} dr < \infty.$$

Denote by $L^2(\nu_\beta^N)$ the Hilbert space of functions f on Ω^N such that $\nu_\beta^N(f^2) < \infty$. S_N is formally symmetric on $L^2(\nu_\beta^N)$ and A_N is formally antisymmetric on $L^2(\nu_\beta^N)$. In fact, it is easy to see that for smooth functions f and g in a core of the operators S_N and A_N, we have for all $\beta > 0$

$$\int_{\mathbb{R}^{2N}} S_N(f) g \, \nu_\beta^N(dpdr) = \int_{\mathbb{R}^{2N}} f S_N(g) \, \nu_\beta^N(dpdr),$$

and

$$\int_{\mathbb{R}^{2N}} A_N(f) g \, \nu_\beta^N(dpdr) = -\int_{\mathbb{R}^{2N}} f A_N(g) \, \nu_\beta^N(dpdr).$$

In particular, the diffusion is invariant with respect to all the measures ν_β^N. The distribution ν_β^N is called canonical Gibbs measure at temperature $T = \beta^{-1}$. Notice that $r_1, ..., r_N, p_1, ..., p_N$ are independently distributed under this probability measure.

On the other hand, for every $\beta > 0$ the Dirichlet form of the diffusion with respect to ν_β^N is given by

$$\mathcal{D}_{N,\beta}(f) = \frac{\gamma}{2}\int_{\mathbb{R}^{2N}} \sum_{i\in\mathbb{T}_N} \{[X_i(f)]^2 + [Y_{i,i+1}(f)]^2\} \nu_\beta^N(dpdr).$$

We will use the notation ν_β for the product measures on the configuration space $\Omega := (\mathbb{R}^2)^{\mathbb{Z}}$, namely on the infinite lattice with marginal given by

$\nu_\beta|_{\{1,2,\dots,N\}} = \nu_\beta^N$. The expectation with respect to ν_β will be sometimes denoted by

$$\int_\Omega f \nu_\beta(dpdr) = \langle f \rangle_\beta.$$

Denote by $\{\omega(t) = (p(t), r(t)); t \geq 0\}$ the Markov process generated by $N^2 L_N^\gamma$ (the factor N^2 corresponds to an acceleration of time). Let $C(\mathbb{R}_+, \Omega^N)$ be the space of continuous trajectories on the configuration space. For any fixed time $T > 0$ and for a given measure μ^N on Ω^N, the probability measure on $C([0,T], \Omega^N)$ induced by this Markov process starting from μ^N will be denoted by $\mathbb{P}_{\mu^N}$. As usual, expectation with respect to $\mathbb{P}_{\mu^N}$ will be denoted by $\mathbb{E}_{\mu^N}$. The diffusion generated by $N^2 L_N^\gamma$ can also be described by the following system of stochastic differential equations

$$\begin{aligned} dp_i(t) &= N^2[V'(r_{i+1}) - V'(r_i) - \frac{\gamma p_i}{2}\{V''(r_i) + V''(r_{i+1})\}]dt \\ &\qquad\qquad + \sqrt{\gamma} N \{V'(r_{i+1}) dB_i^1 - V'(r_i) dB_i^2\}, \\ dr_i(t) &= N^2[p_i - p_{i-1} - \gamma V'(r_i)]dt + \sqrt{\gamma} N\{-p_{i-1} dB_{i-1}^1 + p_i dB_i^2\} \end{aligned}$$

where $\{B_i^1, B_i^2\}_{i\in\mathbb{T}_N}$ are $2N$-independent standard Brownian motions.

Since total energy is conserved, the movement is constrained on the *microcanonical* surface of constant energy

$$\Sigma_{N,E} = \left\{\omega \in \Omega^N; \sum_{i=1}^N \mathcal{E}_i = NE\right\}. \tag{2.1}$$

Our conditions on V assure that these surfaces are always connected. The vector fields $\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ are tangent to this surface, and as we show in Section 13, Lie$\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ generates the all tangent space. Consequently the *microcanonical* measures

$$\nu_{N,E}(\cdot) = \nu_\beta^N(\cdot | \Sigma_{N,E})$$

are ergodic for our dynamics. We could have chosen $\nu_{N,E}$ as initial distribution, but since by the equivalence of ensembles it converges to $\nu_{\beta(E)}$ as $N \to \infty$, it would have been irrelevant. Here $\beta(E)$ is defined as the inverse function of

$$E(\beta) = \int \mathcal{E}_0 d\nu_\beta = \frac{1}{2\beta} + \tilde{V}(\beta) \tag{2.2}$$

where $\tilde{V}(\beta) = \langle V(r_0)\rangle_\beta$.

By Ito's formula, we have

$$d\mathcal{E}_i(t) = N^2[W_{i-1,i} - W_{i,i+1}]dt + N\{\sigma_{i-1,i} dB_{i-1}^1 - \sigma_{i,i+1} dB_i^1\} \tag{2.3}$$

where

$$\begin{aligned} W_{i,i+1} &= W_{i,i+1}^A + W_{i,i+1}^S, \\ W_{i,i+1}^A &= -p_i V'(r_{i+1}), \\ W_{i,i+1}^S &= \frac{\gamma}{2}\{p_i^2 V''(r_{i+1}) - V'(r_{i+1})^2\}, \\ \sigma_{i,i+1} &= -\sqrt{\gamma} p_i V'(r_{i+1}). \end{aligned} \tag{2.4}$$

We can think of $W_{i,i+1}$ as being the instantaneous microscopic current of energy between the oscillator i and the oscillator $i+1$. Observe that the current $W_{i,i+1}$ cannot be written as the gradient of a local function, neither by an exact fluctuation-dissipation equation, i.e. as the sum of a gradient and a dissipative term of the form $L_N^{\gamma}(\tau_i h)$. That means, we are in the nongradient case.

Define the empirical energy distribution associated to the process by

$$\pi_t^N(\omega, du) = \frac{1}{N}\sum_{i\in\mathbb{T}_N} \mathcal{E}_i(t)\delta_{\frac{i}{N}}(du), \quad 0 \le t \le T, \quad u \in \mathbb{T},$$

and $\langle \pi_t^N, f\rangle$ stands for the integration of f with respect to π_t^N.

Notice that we are using here as space variable the *material* coordinate i/N, and not the physical positions q_i. These two descriptions are equivalent but in our model the material (Lagrangian) coordinates simplify notations.

It is easy to prove that, starting with the equilibrium measure ν_β^N (or with $\nu_{N,E(\beta)}$), we have $\pi_t^N \longrightarrow E(\beta)du$ as weak convergence in probability.

We want to investigate the fluctuation of the empirical measure π^N with respect to this limit. Denote by Y_t^N the *empirical energy fluctuation field* acting on a smooth function $H : \mathbb{T} \to \mathbb{R}$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} H(\frac{i}{N})\{\mathcal{E}_i(t) - E(\beta)\}.$$

The limit process will be described by $\{Y_t\}_{t\ge 0}$, the stationary generalized Ornstein-Uhlenbeck process with zero mean and covariances given by

$$\mathbb{E}[Y_t(H_1)Y_0(H_2)] = \frac{\chi(\beta)}{\sqrt{4\pi t D(\beta)}}\iint_{\mathbb{R}^2} dudv \bar{H}_1(u)e^{-\frac{(u-v)^2}{4tD(\beta)}}\bar{H}_2(v)$$

for every $t \ge 0$. Here $\chi(\beta)$ stands for the variance of the energy (the thermal capacity in this context) given by

$$\chi(\beta) = \langle \mathcal{E}_0^2\rangle_\beta - \langle \mathcal{E}_0\rangle_\beta^2 = \frac{1}{2\beta^2} - \tilde{V}'(\beta)$$

and $\bar{H}_1(u)$ (resp. $\bar{H}_2(u)$) is the periodic extension of the smooth function H_1 (resp. H_2) to the real line, and $D(\beta)$ is the diffusion coefficient determined later.

Consider for $k > \frac{5}{2}$ the Sobolev space $\mathfrak{H}_{-k}$ of the distributions Y on $\mathbb{T}$ such that they have finite norm

$$\|Y\|_{-k}^2 = \sum_{n\ge 1}(\pi n)^{-2k}|Y(e_n)|^2$$

with $e_n(x) = \sqrt{2}\sin(\pi n x)$. Denote by $\mathbb{Q}_N$ the probability measure on $C([0,T], \mathfrak{H}_{-k})$ induced by the energy fluctuation field Y_t^N and the Markov process $\{\omega^N(t), t \ge 0\}$ defined at the beginning of this section, starting from

the equilibrium probability measure ν_β^N. Let $\mathbb{Q}$ be the probability measure on the space $C([0,T],\mathfrak{H}_{-k})$ corresponding to the generalized Ornstein-Uhlenbeck process Y_t defined above. We are now ready to state the main result of this work.

Theorem 1. *The sequence of the probability measures $\{\mathbb{Q}_N\}_{N\geq 1}$ converges weakly to the probability measure $\mathbb{Q}$.*

Remark 2.4. *For each $H \in C^\infty(\mathbb{T})$,*

$$M_t^{D,H} := Y_t(H) - Y_0(H) - \int_0^t Y_s(D(\beta)\Delta H)ds, \tag{2.5}$$

and

$$N_t^{D,H} := (M_t^{D,H})^2 - 2t\chi(\beta)D(\beta)\langle (H')^2\rangle_{L^2(\mathbb{T})} \tag{2.6}$$

are $L^1(\mathbb{Q})$-martingales.

3. Strategy of the proof of the main theorem

We follow the argument in Section 11 in [12]. Theorem 1 follows from the following three statements:

(i) $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight,

(ii) the restriction of any limit point $\mathbb{Q}^*$ of a convergent subsequence of $\{\mathbb{Q}_N\}_{N\geq 1}$ to $\mathcal{F}_0$ is Gaussian fields with covariances given by

$$\mathbb{E}[Y(H_1)Y(H_2)] = \chi(\beta)\langle H_1, H_2\rangle_{L^2(\mathbb{T})},$$

(iii) all limit points $\mathbb{Q}^*$ of convergent subsequences of $\{\mathbb{Q}_N\}_{N\geq 1}$ solve the martingale problems (2.5) and (2.6).

The proof of (ii) is obtained by a direct consequence of the central limit theorem for independent variables. We will prove (i) in section 7. We prove here the main point, i.e. (iii).

For a given smooth function $H : \mathbb{T} \to \mathbb{R}$, we begin by rewriting $Y_t^N(H)$ as

$$Y_t^N(H) = Y_0^N(H) + \int_0^t \sqrt{N} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})W_{i,i+1}ds + M^{H,N}(t) \tag{3.1}$$

where $\nabla^N H$ represents the discrete derivative of H:

$$(\nabla^N H)(\frac{i}{N}) = N[H(\frac{i+1}{N}) - H(\frac{i}{N})]$$

and the martingale $M^{H,N}(t)$ is

$$M^{H,N}(t) = \int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})\sigma_{i,i+1}dB_i^1.$$

Define $\mathcal{C}$ the set of all smooth local functions F on $\Omega = (\mathbb{R}^2)^{\mathbb{Z}}$ with compact support, and define the formal sum

$$\Gamma_F = \sum_{j\in\mathbb{Z}} \tau_j F \tag{3.2}$$

where τ_j is the shift on $\mathbb{Z}$. Observe that $X_i\Gamma_F$ and $Y_{i,i+1}\Gamma_F$ are always well-defined.

We want to introduce here a fluctuation-dissipation approximation of the current $W_{i,i+1}$ in $C_\beta(p_{i,i+1}^2 - p_i^2) + L^\gamma \tau_i F$, for a proper constant c_β, and for that pourpouse we can decompose (3.1) with any fixed $F \in \mathcal{C}$ as follows:

$$\begin{aligned} Y_t^N(H) = Y_0^N(H) + \int_0^t Y_s^N(D(\beta)\Delta^N H) ds + M_{N,F,t}^1(H) \\ + I_{N,F,t}^1 + I_{N,F,t}^2 + M_{N,F,t}^2 + D(\beta)\chi(\beta)\beta^2 I_{N,t}^3 \end{aligned} \tag{3.3}$$

where

$$I_{N,F,t}^1(H) = \int_0^t \sqrt{N} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[W_{i,i+1} + D(\beta)\chi(\beta)\beta^2(p_{i+1}^2 - p_i^2) - L_N^\gamma(\tau_i F)]ds,$$

$$I_{N,F,t}^2(H) = \int_0^t \sqrt{N} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N}) L_N^\gamma(\tau_i F) ds,$$

$$I_{N,t}^3(H) = \int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} \Delta^N H(\frac{i}{N})[(p_i^2 - \frac{1}{\beta}) - \frac{1}{\chi(\beta)\beta^2}\{\mathcal{E}_i - E(\beta)\}],$$

$$M_{N,F,t}^1(H) = \int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[(\sigma_{i,i+1} + \sqrt{\gamma} Y_{i,i+1}(\Gamma_F))dB_i^1 - \sqrt{\gamma} X_i(\Gamma_F) dB_i^2],$$

$$M_{N,F,t}^2(H) = \int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[-\sqrt{\gamma} Y_{i,i+1}(\Gamma_F) dB_i^1 + \sqrt{\gamma} X_i(\Gamma_F) dB_i^2].$$

The proof of (iii) is reduced to the following lemmas:

Lemma 3.1. *For every smooth function $H : \mathbb{T} \to \mathbb{R}$ and every function $F \in \mathcal{C}$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\beta^N}\left[\sup_{0\le t\le T}(I_{N,F,t}^2(H) + M_{N,F,t}^2(H))^2\right] = 0.$$

Lemma 3.2. *For every smooth function $H : \mathbb{T} \to \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\beta^N}\left[\sup_{0\le t\le T}(I_{N,t}^3(H))^2\right] = 0.$$

Lemma 3.3. *There exists a sequence of functions $\{F_K\}_{K\in\mathbb{N}} \in \mathcal{C}$ such that, for every smooth function $H : \mathbb{T} \to \mathbb{R}$,*

$$\lim_{K\to\infty}\lim_{N\to\infty} \mathbb{E}_{\nu_\beta^N}\left[\sup_{0\le t\le T}(I_{N,F_K,t}^1(H))^2\right] = 0.$$

Moreover, for this sequence $\{F_K\}_{K\in\mathbb{N}}$,

$$\lim_{K\to\infty} E_{\nu_\beta}[(\sigma_{0,1} + \sqrt{\gamma} Y_{0,1}(\Gamma_{F_K}))^2 + (\sqrt{\gamma} X_0(\Gamma_{F_K}))^2] = 2D(\beta)\chi(\beta) = \frac{2\tilde{D}(\beta)}{\beta^2}$$

where $\tilde{D}(\beta) := D(\beta)\chi(\beta)\beta^2$. Note that

$$I_{N,F,t}^1(H) = \int_0^t \sqrt{N} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[W_{i,i+1} + \tilde{D}(\beta)(p_{i+1}^2 - p_i^2) - L_N(\tau_i F)]ds.$$

As a consequence of Lemma 3.3, the martingale $M^1_{N,F_K,t}(H)$ will converge, as $N \to \infty$ and $K \to \infty$ to a martingale $M_t^{D,H}$ of quadratic variation $2tD(\beta)\chi(\beta)\int_{\mathbb{T}} H'(u)^2 du$, and the limit $Y_t(H)$ of $Y_t^N(H)$ will satisfy the equation

$$Y_t(H) = Y_0(H) + \int_0^t Y_s(D(\beta)\Delta H)\, ds + M_t^{D,H}. \tag{3.4}$$

This martingale problem is solved uniquely by the generalize Ornstein-Uhlenbeck process Y_t defined above.

Now we proceed to give a proof of Lemma 3.1. Lemma 3.2 will be proven in Section 6, while Lemma 3.3 will be the content of the rest of the article.

Proof of Lemma 3.1. Let us define

$$\zeta_{N,F}(t) = \frac{1}{N^{\frac{3}{2}}} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})\tau_i F(\omega_t^N).$$

From Ito's formula, we obtain

$$\begin{aligned}\zeta_{N,F}(t) &= \zeta_{N,F}(0) + I^2_{N,F,t}(H)\\ &\quad+ \int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})\sqrt{\gamma} \sum_{j\in\mathbb{T}_N} [-Y_{j,j+1}(\tau_i F) dB_j^1 + X_j(\tau_i F) dB_j^2].\end{aligned}$$

Therefore,

$$\begin{aligned}&(I^2_{N,F,t}(H) + M^2_{N,F,t}(H))^2 \le 2(\zeta_{N,F}(t) - \zeta_{N,F}(0))^2\\ &+2\left\{\int_0^t \frac{1}{\sqrt{N}} \sum_{i,j\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})\sqrt{\gamma}[-Y_{j,j+1}(\tau_i F)dB_j^1 + X_j(\tau_i F)dB_j^2] - M^2_{N,F,t}(H)\right\}^2.\end{aligned}$$

Since F is bounded and H is smooth, it is easy to see that the first term is (uniformly) of order $\frac{1}{N}$. Using additionally the conditions on F, and Doob's inequality we can prove that the expectation of the second term is also of order $\frac{1}{N}$. □

The proof of Lemma 3.1 is the hard part of the paper. We will need some tools to estimate space-time variances through variational formulas, as explained in the next section. In order to establish these variational formula we need some finite dimensional approximations of the solutions, see section 8, where a bound on the spectral gap of S is needed. This is proven in section 12.

4. Space-Time Variance bounds

In the following we will simply denote by $\langle\cdot\rangle$ the expectation with respect to the grand-canonical measure ν_β.

In order to prove lemmas 3.2 and 3.3, we will make use of the following general bound for time variances:

Proposition 4.1. *Let F be a smooth function in $L^2(\nu_\beta^N)$ satisfying $E_{\nu_{N,E}}[F] = 0$ for all $E>0$. Then*

$$\mathbb{E}_{\nu_\beta^N}\left(\sup_{0\le t\le T}\left[\int_0^t F(\omega_s^N)ds\right]^2\right) \;\le\; \frac{16T}{\gamma N^2}\langle F\,(-S_N)^{-1}\,F\rangle. \tag{4.1}$$

A proof of (4.1) can be found in [7] or in [14].

Observe that, by the spectral gap for S_N proven in Section 12, the right hand side is always well defined. We want to use the bound (4.1) for functions of the type $F = \sum_j G(j/N)\tau_j\phi$, for a certain class of local functions ϕ.

First, we introduce some notations. We denote $\tilde{\mathcal{C}}$ the set of smooth local functions f on $\Omega = (\mathbb{R}^2)^{\mathbb{Z}}$ satisfying that

$$f\in L^2(\nu_\beta),\;\; X_i f(p,r)\in L^2(\nu_\beta),\;\; Y_{i,i+1}f\in L^2(\nu_\beta)$$

for all $i\in\mathbb{Z}$. Note that $\mathcal{C}\subset\tilde{\mathcal{C}}$. Here and after, we consider operators L^γ, S^γ and A acting on functions f in $\mathcal{C}$ as

$$L^\gamma f = S^\gamma f + Af,\quad S^\gamma f = \frac{\gamma}{2}\sum_{i\in\mathbb{Z}}\{(X_i)^2 f + (Y_{i,i+1})^2 f\},\quad Af = \sum_{i\in\mathbb{Z}} X_i f - Y_{i,i+1}f.$$

For a fixed positive integer l, we define $\Lambda_l := \{-l,-l+1,\dots,l-1,l\}$ and $L^\gamma_{\Lambda_l}$, $S^\gamma_{\Lambda_l}$ the restriction of the generator L^γ, S^γ to Λ_l respectively. For Ψ in $\mathcal{C}$, denote by s_Ψ the smallest positive integer s such that Λ_s contains the support of Ψ. Let $\mathcal{C}_0$ be a subspace of local functions defined as follows:

$$\mathcal{C}_0 = \{\,f; f = \sum_{i\in\Lambda}[X_i(F_i)+Y_{i,i+1}(G_i)] \text{ for some } \Lambda\subset\subset\mathbb{Z} \text{ and} \{F_i\}_{i\in\Lambda},\{G_i\}_{i\in\Lambda}\in\tilde{\mathcal{C}}\}.$$

First, we note some useful properties of the space $\mathcal{C}_0$.

Lemma 4.1. *(i) For any $f\in\mathcal{C}_0$, $l\ge s_f+1$ and $E>0$, $E_{\nu_{l,E}}[f]=0$.*
(ii) $W^S_{0,1}$, $W^A_{0,1}$ and $p_1^2-p_0^2$ are elements of $\mathcal{C}_0$.
(iii) For any $F\in\mathcal{C}$, $L^\gamma F$, $S^\gamma F$ and AF are elements of $\mathcal{C}_0$.

Proof. (i) and (iii) are straightforward.
(ii): We have

$$\begin{aligned}
W^S_{0,1} &= \frac{\gamma}{2}\{p_0^2V''(r_1) - V'(r_1)^2\} = \frac{\gamma}{2}Y_{0,1}(p_0V'(r_1))\\
W^A_{0,1} &= -p_0V'(r_1) = Y_{0,1}(-V(r_1))\\
p_1^2-p_0^2 &= X_1\{(p_0+p_1)r_1\} - Y_{0,1}\{(p_0+p_1)r_1\}.
\end{aligned}$$

□

Next, we study the variance

$$(2l)^{-1}\langle(-S^\gamma_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi,\sum_{|i|\le l_\psi}\tau_i\psi\rangle$$

for $\psi\in\mathcal{C}_0$ where $l_\psi = l-s_\psi-1$. We start with introducing a semi-norm on $\mathcal{C}_0$, which is closely related to the central limit theorem variance. For cylinder

functions g, h in $\mathcal{C}_0$, let

$$\ll g,h \gg_{*} = \sum_{i\in\mathbb{Z}} \langle g, \tau_i h\rangle \quad \text{and} \quad \ll g \gg_{**} = \sum_{i\in\mathbb{Z}} i\langle g, \mathcal{E}_i\rangle. \tag{4.2}$$

$\ll g,h \gg_{*}$ and $\ll g \gg_{**}$ are well defined because g and h belong to $\mathcal{C}_0$ and therefore all but a finite number of terms vanish.

Notice that if $g = \sum_{i\in\Lambda} X_i F_i + Y_{i,i+1}G_i$ then we can compute

$$\begin{aligned} \ll g \gg_{**} &= \lim_{l\to\infty} -\sum_{i\in\Lambda}\sum_{j=-l}^{l} j\langle G_i, Y_{i,i+1}\mathcal{E}_j\rangle \\ &= -\sum_{i\in\Lambda}\left(i\langle G_i, Y_{i,i+1}\mathcal{E}_i\rangle + (i+1)\langle G_i, Y_{i,i+1}\mathcal{E}_{i+1}\rangle\right) \\ &= -\sum_{i\in\Lambda}\langle p_i V'(r_{i+1})G_i\rangle. \end{aligned}$$

For h in $\mathcal{C}_0$, define the semi-norm $\|h\|_{-1}$ by

$$\begin{aligned} \|h\|_{-1}^2 = \sup_{g\in\mathcal{C}, a\in\mathbb{R}} \{ & 2 \ll g,h \gg_{*} + 2a \ll h \gg_{**} \\ & - \frac{\gamma}{2}\langle (ap_0V'(r_1) + Y_{0,1}\Gamma_g)^2\rangle - \frac{\gamma}{2}\langle (X_0\Gamma_g)^2\rangle\}. \end{aligned} \tag{4.3}$$

We investigate several properties of the semi-norm $\|\cdot\|_{-1}$ in the next section, while in this section we prove the following key proposition:

Proposition 4.2. *Consider a local function ψ in $\mathcal{C}_0$. Then,*

$$\lim_{l\to\infty}(2l)^{-1}\langle (-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle = \|\psi\|_{-1}^2.$$

The proof is divided into two lemmas.

Lemma 4.2. *For ψ in $\mathcal{C}_0$*

$$\liminf_{N\to\infty}(2l)^{-1}\langle (-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle \ge \|\psi\|_{-1}^2.$$

Lemma 4.3. *For ψ in $\mathcal{C}_0$*

$$\limsup_{N\to\infty}(2l)^{-1}\langle (-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle \le \|\psi\|_{-1}^2.$$

Proof of Lemma 4.2. Define

$$A_l := \sum_{i=-l}^{l-1}\tau_i W^S_{0,1}$$

and for $F\in\mathcal{C}$, let

$$H^F_l := \sum_{|i|\le l-s_F-1}\tau_i S^{\gamma}F.$$

It is easy to see that

$$\lim_{N\to\infty}(2l)^{-1}\langle (-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, A_l\rangle = - \ll \psi \gg_{**}, \tag{4.4}$$

$$\lim_{l\to\infty}(2l)^{-1}\langle (-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, H^F_l\rangle = - \ll \psi, F \gg_{*}, \tag{4.5}$$

$$\lim_{l\to\infty}(2l)^{-1}\langle(-S^{\gamma}_{\Lambda_l})^{-1}(aA_l + H^F_l), aA_l + H^F_l\rangle$$
$$= \frac{\gamma}{2}\langle(ap_0V'(r_1) + Y_{0,1}\Gamma_F)^2\rangle + \frac{\gamma}{2}\langle(X_0\Gamma_F)^2\rangle. \qquad (4.6)$$

We just prove here (4.4), the other relations are proven in similar way. Assume for the simplicity that $\psi = X_0F + Y_{0,1}G$, the general case follows by linearity. Since $A_l = S^{\gamma}_{\Lambda_l}\sum_{j=-l}^{l} j\mathcal{E}_j$

$$\begin{aligned}(2l)^{-1}\langle(-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, A_l\rangle &= -(2l)^{-1}\sum_{j=-l}^{l}\sum_{|i|\le l_\psi} j\langle\psi,\mathcal{E}_{j-i}\rangle\\ &= (2l)^{-1}\sum_{|i|\le l_\psi}\sum_{j=-l}^{l} j\langle G, Y_{0,1}\mathcal{E}_{j-i}\rangle\\ &= (2l)^{-1}\sum_{|i|\le l_\psi}(i\langle G, Y_{0,1}\mathcal{E}_0\rangle + (i+1)\langle G, Y_{0,1}\mathcal{E}_1\rangle)\\ &= (2l)^{-1}(2l_\psi+1)\langle G, p_0V'(r_1)\rangle \underset{l\to\infty}{\longrightarrow} - \ll\psi\gg_{**}\end{aligned}$$

Then, obviously,

$$\begin{aligned}&\liminf_{l\to\infty}(2l)^{-1}\langle(-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle\\ &\quad\ge \liminf_{l\to\infty}[2(2l)^{-1}\langle(-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, -(aA_l + H^F_l)\rangle\\ &\qquad - (2l)^{-1}\langle(-S^{\gamma}_{\Lambda_l})^{-1}(aA_l + H^F_l), (aA_l + H^F_l)\rangle]\\ &= 2\ll\psi, F\gg_* + 2a\ll\psi\gg_{**} - \frac{\gamma}{2}\langle(ap_0V'(r_1) + Y_{0,1}\Gamma_F)^2\rangle - \frac{\gamma}{2}\langle(X_0\Gamma_F)^2\rangle.\end{aligned}$$

Then, taking the supremum of $a \in \mathbb{R}$ and $F \in \mathcal{C}$ we obtain the desired inequality. □

Proof of Lemma 4.3. Let us assume for simplicity of notation that $\psi = X_0F + Y_{0,1}G$. The general case will follow straightforwardly. We use the variational formula

$$\begin{aligned}&(2l)^{-1}\langle(-S^{\gamma}_{\Lambda_l})^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle\\ &\quad= \sup_h\left\{2\langle\psi, \frac{1}{2l}\sum_{|i|\le l_\psi}\tau_i h\rangle - \frac{\gamma}{4l}\mathcal{D}_l(h)\right\}\\ &\quad= \sup_h\left\{2\langle FX_0(\frac{1}{2l}\sum_{|i|\le l_\psi}\tau_i h) + GY_{0,1}(\frac{1}{2l}\sum_{|i|\le l_\psi}\tau_i h)\rangle - \frac{\gamma}{4l}\mathcal{D}_l(h)\right\}\end{aligned}$$

where

$$\mathcal{D}_l(h) = \sum_{|i|\le l}\langle(X_ih)^2\rangle + \sum_{i=-l}^{l-1}\langle(Y_{i,i+1}h)^2\rangle.$$

The supremum can be restrained in the class of functions h that are localized in Λ_l and such that $\mathcal{D}_l(h) \le C_\psi l$.

Notice that

$$\left|\langle F X_0(\frac{1}{2l}\sum_{l_\psi\le|i|\le l}\tau_i h) + G Y_{0,1}(\frac{1}{2l}\sum_{l_\psi\le|i|\le l}\tau_i h)\rangle\right| \le \frac{C_\psi}{2l}\mathcal{D}_l(h)^{1/2}$$

so, calling

$$\xi_0^l(h) = X_0(\frac{1}{2l}\sum_{|i|\le l}\tau_i h), \qquad \xi_1^l(h) = Y_{0,1}(\frac{1}{2l}\sum_{|i|\le l}\tau_i h)$$

and observing that, by Schwarz inequality

$$\langle (\xi_0^l(h))^2 + (\xi_1^l(h))^2\rangle \ \le \frac{1}{2l}\mathcal{D}_l(h)$$

we obtain the upper bound

$$\begin{aligned}&(2l)^{-1}\langle(-S_{\Lambda_l}^\gamma)^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle\\ &\qquad\le \sup_h\Big\{2\langle F,\xi_0^l(h)\rangle + 2\langle G,\xi_1^l(h)\rangle - \frac{\gamma}{2}\left(\langle(\xi_0^l(h))^2 + (\xi_1^l(h))^2\rangle\right)\Big\} + C_\psi l^{-1/2}.\end{aligned}$$

Since for any choice of a sequence $\{h_l\}_l$ satisfying $\mathcal{D}_l(h_l) \le C_\psi l$, we have that the sequence $(\xi_0^l(h_l),\xi_1^l(h_l))$ is uniformly bounded in $L^2(\nu_\beta)$, we can extract convergent subsequences in $L^2(\nu_\beta)$. All limit vectors (ξ_0,ξ_1) that we obtain as limit points of $(\xi_0^l(h_l),\xi_1^l(h_l))$ are *closed* in the sense specified in Section 9. We call this set of closed functions $\mathfrak{h}_c$, and we have obtained that

$$\begin{aligned}&\limsup_{l\to\infty}(2l)^{-1}\langle(-S_{\Lambda_l}^\gamma)^{-1}\sum_{|i|\le l_\psi}\tau_i\psi, \sum_{|i|\le l_\psi}\tau_i\psi\rangle\\ &\qquad\qquad\le \sup_{(\xi_0,\xi_1)\in\mathfrak{h}_c}\Big\{2\langle F,\xi_0\rangle + 2\langle G,\xi_1\rangle - \frac{\gamma}{2}\left(\langle(\xi_0)^2 + (\xi_1)^2\rangle\right)\Big\}\end{aligned}$$

and the desired upper bound follows from the characterization of $\mathfrak{h}_c$ proved by Theorem 3 in Section 9. □

We are now in the position to state the main result of this section:

Theorem 2. *Let $\psi\in\mathcal{C}_0$, and G a smooth function on $\mathbb{T}$. Then*

$$\limsup_{N\to\infty}\mathbb{E}_{\nu_\beta^N}\left(\sup_{0\le t\le T}\left[N^{1/2}\int_0^t\sum_{i\in\mathbb{T}_N}G(i/N)\tau_i\psi(\omega_s)ds\right]^2\right) \le \frac{CT}{\gamma}\|\psi\|_{-1}^2\int_{\mathbb{T}}G(u)^2du. \tag{4.7}$$

Proof of Theorem 2. We follow the argument in [7], Theorem 4.2.

First we prove the simpler bound

$$\mathbb{E}_{\nu_\beta^N}\left(\sup_{0\le t\le T}\left[N^{1/2}\int_0^t\sum_{i\in\mathbb{T}_N}G(i/N)\tau_i\psi(\omega_s)ds\right]^2\right) \le \frac{C_\psi T}{\gamma}\frac{1}{N}\sum_{i\in\mathbb{T}_N}G(i/N)^2 \tag{4.8}$$

for some finite constant C_ψ.

By (4.1), the left side of (4.8) is bounded by

$$16T\langle N^{1/2}\sum_{i\in\mathbb{T}_N} G(i/N)\tau_i\psi, (-N^2\gamma S_N)^{-1}N^{1/2}\sum_{i\in\mathbb{T}_N} G(i/N)\tau_i\psi\rangle$$

that can be written with the variational formula

$$16T\sup_f\left\{N^{1/2}\sum_{i\in\mathbb{T}_N} G(i/N)\langle f\tau_i\psi\rangle - N^2\gamma\langle f,(-S_N)f\rangle\right\}.$$

Since $\psi\in\mathcal{C}_0$, there exists Ψ_j and Φ_j belonging to $\mathcal{C}$, for $j\in\Lambda\subset\subset\mathbb{Z}$, such that $\psi=\sum_{j\in\Lambda}\left[X_j(\Psi_j)+Y_{j,j+1}(\Phi_j)\right]$, and we can bound by integration by parts

$$\langle f\tau_i\psi\rangle\le C_\psi\langle\sum_{j\in\Lambda}[(X_{-i-j}f)^2+(Y_{-i-j,-i-j+1}f)^2]\rangle^{1/2}$$

and again by Schwarz inequality

$$N^{1/2}\sum_{i\in\mathbb{T}_N} G(i/N)\langle f\tau_i\psi\rangle\le\left(\frac{1}{N}\sum_{i\in\mathbb{T}_N} G(i/N)^2\right)^{1/2}\left(N^2C_\psi\langle f,(-S_N)f\rangle\right)^{1/2}$$

and maximizing on f we obtain (4.8).

Now we have to refine the bound by showing that the constant on the right hand side is proportional to $\|\psi\|_{-1}^2$. In order to do this, we have to perform a further microscopic average: given $K<<N$, in (4.7) we want to substitute

$$\sqrt{N}\sum_{i\in\mathbb{T}_N} G(i/N)\tau_i\psi$$

with

$$\sqrt{N}\sum_{j\in\mathbb{T}_N} G(j/N)\frac{1}{2K+1}\sum_{|i-j|\le K}\tau_i\psi.$$

Then the difference is estimated by

$$\mathbb{E}_{\nu_\beta^N}\left(\sup_{0\le t\le T}\left[N^{1/2}\int_0^t\sum_{i,j\in\mathbb{T}_N,|i-j|\le K}\frac{1}{2K+1}(G(i/N)-G(j/N))\tau_i\psi(\omega_s)ds\right]^2\right)$$

that by (4.8) is bounded by C_GK/N and tends to 0 as $N\to\infty$.

So we are left with

$$\mathbb{E}_{\nu_\beta^N}\left(\sup_{0\le t\le T}\left[\frac{N^{1/2}}{2K+1}\int_0^t\sum_{j\in\mathbb{T}_N} G(j/N)\tau_j\hat\psi_K(\omega_s)ds\right]^2\right)$$

where $\hat\psi_K=\sum_{|i|\le K}\tau_i\psi$. By (4.1) this is bounded by

$$\frac{CT}{2K+1}\sum_{j\in\mathbb{T}_N}\sup_f\left\{\sqrt{N}G(j/N)\langle\tau_j\hat\psi_K f\rangle - N^2\frac{\gamma}{2}\sum_{|i-j|\le K}\langle(X_if)^2+(Y_{i,i+1}f)^2\rangle\right\}$$
$$\le\frac{CT}{\gamma N}\sum_{j\in\mathbb{T}_N} G(j/N)^2\frac{1}{2K+1}\langle\hat\psi_K,(-S_{\Lambda_K})^{-1}\hat\psi_K\rangle.$$

Taking the limit as $N\to\infty$ and $K\to\infty$ we obtain (4.7). □

Applying Theorem 2 to $I^1_{N,F,t}(H)$ we have

$$\limsup_{N\to\infty} \mathbb{E}_{\nu_\beta^N}\left(\sup_{0\le t\le T}(I^1_{N,F,t}(H))^2\right) \le \frac{CT}{\gamma}\|W_{0,1}+\tilde D(\beta)(p_1^2-p_0^2)-L^\gamma F\|_{-1}^2\int_{\mathbb{T}} H'(u)^2 du. \tag{4.9}$$

To conclude the proof of Lemma 3.3, we need to show that there exists a sequence of local functions F_K in $\mathcal{C}$ such that

$$\|W_{0,1}+\tilde D(\beta)(p_1^2-p_0^2)-L^\gamma F_K\|_{-1}\to 0$$

as $K\to\infty$.

Dear Stefano

I have been thinking about what you wrote for two days. Now I might be very confusing, but I think though it clear that $<U,S_{\Lambda_K}U>=<U,(\sum_{i=-K}^K X_i^2+\sum_{i=-K}^{K-1}Y_i^2)U>$ but it is not clear $<U,S_{\Lambda_K}U>=(\sum_{i=-K}^K<U,X_i^2U>+\sum_{i=-K}^{K-1}<U,Y_i^2U>$ or $<U,S_{\Lambda_K}U>=\sum_{i=-K}^K<(X_iU)^2>+\sum_{i=-K}^{K-1}<(Y_iU)^2>$ where $<,>$ is the inner product in $L^2(\nu_\beta)$. Because we do not know $X_i(X_iU)\in L^2(\nu_\beta)$ or not for each i. I believe there may be some nice proof of them, but I could not find so far. Actually, for any $g\in\mathcal{C}_0$ with $g=\sum(X_iF_i+Y_iG_i)$, we can use the equations $<g,h>=\sum<X_iF_i,h>+<Y_iG_i,h>=\sum<F_i,X_ih>+<G_i,Y_ih>$ because we assume that $X_iF_i,Y_iG_i,F_i,G_i\in L^2$ where h is any smooth function with compact support.

5. Hilbert space

In this section, to prove the first statement of Lemma 3.3, we investigate the properties of the semi norm $\|\cdot\|_{-1}$ introduced in the previous section and the structure of the Hilbert space that it generates.

We first define from $\|\cdot\|_{-1}$ a semi-inner product on $\mathcal{C}_0$ through polarization:

$$\ll g,h\gg_{-1}=\frac{1}{4}\{\|g+h\|_{-1}^2-\|g-h\|_{-1}^2\}. \tag{5.1}$$

It is easy to check that (5.1) defines a semi-inner product on $\mathcal{C}_0$. Denote by $\mathcal{N}$ the kernel of the semi-norm $\|\cdot\|_{-1}$ on $\mathcal{C}_0$. Since $\ll\cdot,\cdot\gg_{-1}$ is a semi-inner product on $\mathcal{C}_0$, the completion of $\mathcal{C}_0|_{\mathcal{N}}$, denoted by $\mathcal{H}_{-1}$, is a Hilbert space.

In the following, in order to simplify notations, we will set $L=L^\gamma$ and $S=S^\gamma$.

By Lemma 4.1, the linear space generated by $W_{0,1}^S$ and $S\mathcal{C}:=\{Sg;\ g\in\mathcal{C}\}$ are subsets of $\mathcal{C}_0$. The first main result of this section consists in showing that $\mathcal{H}_{-1}$ is the completion of $S\mathcal{C}|_{\mathcal{N}}+\{W_{0,1}^S\}$, in other words, that all elements of $\mathcal{H}_{-1}$ can be approximated by $aW_{0,1}^S+Sg$ for some a in $\mathbb{R}$ and g in $\mathcal{C}$. To prove this result we derive two elementary identities:

$$\ll h,Sg\gg_{-1}=-\ll h,g\gg_*,\quad \ll h,W_{0,1}^S\gg_{-1}=-\ll h\gg_{**} \tag{5.2}$$

for all h in $\mathcal{C}_0$ and g in $\mathcal{C}$.

By Proposition 4.2 and (5.1), the semi-inner product $\ll h, g \gg_{-1}$ is the limit of the covariance $(2N)^{-1}\langle(-S_{\Lambda_N})^{-1}\sum_{|i|\le N_g}\tau_i g, \sum_{|i|\le N_h}\tau_i h\rangle$ as $N\uparrow\infty$. In particular, if $g = Sg_0$, for some g_0 in $\mathcal{C}$, the inverse of the operator S cancels with the operator S. Therefore

$$\ll h, Sg_0 \gg_{-1} = -\lim_{N\to\infty}(2N)^{-1}\langle\sum_{|i|\le N_{g_0}}\tau_i g_0, \sum_{|i|\le N_h}\tau_i h\rangle = - \ll g_0, h \gg_* .$$

The second identity is proved by similar way with the elementary relation

$$S_{\Lambda_N}\sum_{i\in\Lambda_N} i\mathcal{E}_i = \sum_{i,i+1\in\Lambda_N} W^S_{i,i+1}.$$

The identities of (5.2) permit to compute the following elementary relations

$$\begin{aligned}\ll W^S_{0,1}, Sh \gg_{-1} &= -\sum_{i\in\mathbb{Z}} i\langle\mathcal{E}_i Sh\rangle = \gamma\langle p_0 V'(r_1)Y_{0,1}\Gamma_h\rangle,\\ \ll p_1^2 - p_0^2, Sh \gg_{-1} &= 0\end{aligned}$$

for all $h\in\mathcal{C}$, and

$$\ll W^S_{0,1}, W^S_{0,1} \gg_{-1} = \frac{\gamma}{2}\langle(p_0V'(r_1))^2\rangle, \quad \ll W^S_{0,1}, p_1^2 - p_0^2 \gg_{-1} = -\frac{1}{\beta^2}.$$

Furthermore,

$$\|aW^S_{0,1} + Sg\|^2_{-1} = \frac{\gamma}{2}\langle(ap_0V'(r_1) + Y_{0,1}\Gamma_g)^2\rangle + \frac{\gamma}{2}\langle(X_0\Gamma_g)^2\rangle$$

for a in $\mathbb{R}$ and g in $\mathcal{C}$. In particular, the variational formula (4.3) for $\|h\|^2_{-1}$ is reduced to the expression

$$\|h\|^2_{-1} = \sup_{a\in\mathbb{R}, g\in\mathcal{C}}\{-2 \ll h, aW^S_{0,1} + Sg \gg_{-1} - \|aW^S_{0,1} + Sg\|^2_{-1}\}. \tag{5.3}$$

Proposition 5.1. *Recall that we denote by $S\mathcal{C}$ the space $\{Sg;\ g\in\mathcal{C}\}$. Then we have*

$$\mathcal{H}_{-1} = \overline{S\mathcal{C}}|_{\mathcal{N}} + \{W^S_{0,1}\}.$$

Proof. The inclusion $\mathcal{H}_{-1} \supset \overline{S\mathcal{C}}|_{\mathcal{N}} + \{W^S_{0,1}\}$ is obvious. Then we have only to show that if $h\in\mathcal{H}_{-1}$ such that $\ll h, W^S_{0,1} \gg = 0$ and $\ll h, Sg \gg = 0$ for all $g\in\mathcal{C}$, then $\|h\|_{-1} = 0$. This follows directly from the variational formula (5.3). □

Corollary 5.1. *We have*

$$\mathcal{H}_{-1} = \overline{S\mathcal{C}}|_{\mathcal{N}} \oplus \{W^{0,1}_S\} = \overline{S\mathcal{C}}|_{\mathcal{N}} \oplus \{p_1^2 - p_0^2\}.$$

Proof. Since $\ll p_1^2 - p_0^2, Sh \gg_{-1} = 0$ for all $h\in\mathcal{C}$ and $\ll W^S_{0,1}, p_1^2 - p_0^2 \gg_{-1} = -\frac{1}{\beta^2}$, the result follows from Proposition 5.1 straightforwardly. □

Remark 5.1. *While the statement of Proposition 5.1 claims that $\mathcal{H}_{-1}$ is generated by the spaces $S\mathcal{C}$ and $\{W^S_{0,1}\}(=\{aW^S_{0,1}; a\in\mathbb{R}\})$, the statement of Corollary 5.1 claims also that the intersection of them is the trivial set. Note that $S\mathcal{C}$ and $\{W^S_{0,1}\}$ are not orthogonal.*

Next, to replace the space $S\mathcal{C}$ by $L\mathcal{C}$, we show some useful lemmas.

Lemma 5.1. *For all $g, h \in \mathcal{C}$, $\ll Sg, Ah \gg_{-1} = - \ll Ag, Sh \gg_{-1}$. Especially, $\ll Sg, Ag \gg_{-1} = 0$.*

Proof. By the first identity of (5.2),

$$\begin{aligned}
\ll Sg, Ah \gg_{-1} &= - \ll g, Ah \gg_{*} = -\sum_{i\in\mathbb{Z}} \langle \tau_i g, Ah \rangle \\
&= \sum_{i\in\mathbb{Z}} \langle A\tau_i g, h \rangle = \sum_{i\in\mathbb{Z}} \langle \tau_i Ag, h \rangle \\
&= \sum_{i\in\mathbb{Z}} \langle Ag, \tau_{-i} h \rangle = \sum_{i\in\mathbb{Z}} \langle Ag, \tau_i h \rangle = - \ll Ag, Sh \gg_{-1} .
\end{aligned}$$

□

Lemma 5.2. *For all $g \in \mathcal{C}$, $\ll Sg, W^A_{0,1} \gg_{-1} = - \ll Ag, W^S_{0,1} \gg_{-1}$.*

Proof. By the first identity of (5.2),

$$\begin{aligned}
\ll Sg, W^A_{0,1} \gg_{-1} &= - \ll g, W^A_{0,1} \gg_{*} = -\sum_{i\in\mathbb{Z}} \langle \tau_i g, W^A_{0,1} \rangle \\
&= -\sum_{i\in\mathbb{Z}} \langle g, W^A_{i,i+1} \rangle = -\sum_{i\in\mathbb{Z}} i \langle g, W^A_{i-1,i} - W^A_{i,i+1} \rangle \\
&= -\sum_{i\in\mathbb{Z}} i \langle g, A\mathcal{E}_i \rangle = \sum_{i\in\mathbb{Z}} i \langle Ag, \mathcal{E}_i \rangle = - \ll Ag, W^S_{0,1} \gg_{-1} .
\end{aligned}$$

□

Lemma 5.3. *For all $a \in \mathbb{R}$ and $g \in \mathcal{C}$,*

$$\ll aW^S_{0,1} + Sg, aW^A_{0,1} + Ag \gg_{-1} = 0.$$

Proof. By the second identity of (5.2), it is easy to see that $\ll W^S_{0,1}, W^A_{0,1} \gg_{-1} = 0$. Then, Lemma 5.1 and Lemma 5.2 conclude the proof straightforwardly. □

Proposition 5.2. *There exists a positive constant C such that for all $g \in \mathcal{C}$,*

$$\|Ag\|^2_{-1} \leq C \|Sg\|^2_{-1}.$$

Proof. By Proposition 5.1, we have the following variational formula for $\|Ag\|^2_{-1}$,

$$\begin{aligned}
\|Ag\|^2_{-1} &= \sup_{a\in\mathbb{R}, h\in\mathcal{C}} \frac{\ll Ag, aW^S_{0,1} + Sh \gg^2_{-1}}{\|aW^S_{0,1} + Sh\|^2_{-1}} \\
&= \max\left\{ \sup_{h\in\mathcal{C}} \frac{\ll Ag, Sh \gg^2_{-1}}{\|Sh\|^2_{-1}}, \sup_{a\neq 0, h\in\mathcal{C}} \frac{\ll Ag, aW^S_{0,1} + Sh \gg^2_{-1}}{\|aW^S_{0,1} + Sh\|^2_{-1}} \right\} \\
&= \max\left\{ \sup_{h\in\mathcal{C}} \frac{\ll Ag, Sh \gg^2_{-1}}{\|Sh\|^2_{-1}}, \sup_{h\in\mathcal{C}} \frac{\ll Ag, W^S_{0,1} + Sh \gg^2_{-1}}{\|W^S_{0,1} + Sh\|^2_{-1}} \right\}.
\end{aligned}$$

By Lemma 8.2 in Section 8, there exists a positive constant C such that $\ll Ag, Sh \gg_{-1}^2 \le C\|Sh\|_{-1}^2\|Sg\|_{-1}^2$ for all $g, h \in \mathcal{C}$. Therefore, we have

$$\sup_{h\in\mathcal{C}} \frac{\ll Ag, Sh \gg_{-1}^2}{\|Sh\|_{-1}^2} \le C\|Sg\|_{-1}^2.$$

On the other hand, by Lemma 5.2, we have

$$\ll Ag, W_{0,1}^S \gg_{-1}^2 = \ll Sg, W_{0,1}^A \gg_{-1}^2 \le \|Sg\|_{-1}^2\|W_{0,1}^A\|_{-1}^2.$$

Therefore,

$$\sup_{h\in\mathcal{C}} \frac{\ll Ag, W_{0,1}^S + Sh \gg_{-1}^2}{\|W_{0,1}^S + Sh\|_{-1}^2} \le \|Sg\|_{-1}^2 \sup_{h\in\mathcal{C}} \frac{2\|W_{0,1}^A\|_{-1}^2 + 2C\|Sh\|_{-1}^2}{\|W_{0,1}^S + Sh\|_{-1}^2}.$$

Now, we only have to show that

$$\sup_{h\in\mathcal{C}} \frac{1}{\|W_{0,1}^S + Sh\|_{-1}} < \infty, \qquad \sup_{h\in\mathcal{C}} \frac{\|Sh\|_{-1}}{\|W_{0,1}^S + Sh\|_{-1}} < \infty.$$

The first inequality follows from Corollary 5.1. To prove the second identity, since we have the first inequality, it is enough to show that

$$\sup_{\substack{t\ge 2, h\in\mathcal{C} \\ \|Sh\|_{-1}^2 = t\|W_{0,1}^S\|_{-1}^2}} \left\{ \frac{\|Sh\|_{-1}}{\|W_{0,1}^S + Sh\|_{-1}} \right\} < \infty.$$

The triangle inequality shows that

$$\|W_{0,1}^S + Sh\|_{-1} \ge \|Sh\|_{-1} - \|W_{0,1}^S\|_{-1} = (\sqrt{t} - 1)\|W_{0,1}^S\|_{-1}$$

for any h satisfying $\|Sh\|_{-1}^2 = t\|W_{0,1}^S\|_{-1}^2$. Then, we obtain that

$$\sup_{\substack{t\ge 2, h\in\mathcal{C} \\ \|Sh\|_{-1}^2 = t\|W_{0,1}^S\|_{-1}^2}} \left\{ \frac{\|Sh\|_{-1}}{\|W_{0,1}^S + Sh\|_{-1}} \right\} \le \sup_{t\ge 2} \left\{ \frac{t}{(\sqrt{t} - 1)^2} \right\} < \infty.$$

□

Now, we have all elements to show the desired decomposition of the Hilbert spaces $\mathcal{H}_{-1}$.

Proposition 5.3. *Denote by $L\mathcal{C}$ the space $\{Lg;\ g \in \mathcal{C}\}$. Then, we have*

$$\mathcal{H}_{-1} = \overline{L\mathcal{C}}|_{\mathcal{N}} + \{p_1^2 - p_0^2\}.$$

Proof. Since $\{p_1^2 - p_0^2\}$ and $L\mathcal{C}$ are contained in $\mathcal{C}_0$ by definition, $\mathcal{H}_{-1}$ contains the right hand space. To prove the converse inclusion, let $h \in \mathcal{H}_{-1}$ so that $\ll h, p_1^2 - p_0^2 \gg_{-1} = 0$ and $\ll h, Lg \gg_{-1} = 0$ for all $g \in \mathcal{C}$. By Corollary 5.1, $h = \lim_{k\to\infty} S^\gamma g_k$ in $\mathcal{H}_{-1}$ for some sequence $g_k \in \mathcal{C}$. Namely,

$$\|h\|_{-1}^2 = \lim_{k\to\infty} \ll Sg_k, Sg_k \gg_{-1} = \lim_{k\to\infty} \ll Sg_k, Lg_k \gg_{-1}$$

since $\ll Sg_k, Ag_k \gg_{-1} = 0$ by Lemma 5.1. On the other hand, by the assumption $\ll h, Lg_k \gg_{-1} = 0$ for all k. Also, by Proposition 5.2,

$$\sup_k \|Lg_k\|_{-1} \le (C + 1) \sup_k \|Sg_k\|_{-1} := C_h$$

is finite. Therefore,

$$\|h\|_{-1}^2 = \lim_{k\to\infty} \ll Sg_k, Lg_k \gg_{-1} = \lim_{k\to\infty} \ll Sg_k - h, Lg_k \gg_{-1}$$
$$\le \lim_{k\to\infty} C_h \|Sg_k - h\|_{-1} = 0.$$

This concludes the proof. □

Lemma 5.4. *We have*

$$\mathcal{H}_{-1} = \overline{L\mathcal{C}}|_{\mathcal{N}} \oplus \{p_1^2 - p_0^2\}.$$

Proof. Let a sequence $g_k \in \mathcal{C}$ satisfy $\lim_{k\to\infty} Lg_k = a(p_1^2 - p_0^2)$ in $\mathcal{H}_{-1}$ for some $a \in \mathbb{R}$. By a similar argument of the proof of Proposition 5.3,

$$\limsup_{k\to\infty} \ll Sg_k, Sg_k \gg_{-1} = \limsup_{k\to\infty} \ll Lg_k, Sg_k \gg_{-1}$$
$$= \limsup_{k\to\infty} \ll Lg_k - a(p_1^2 - p_0^2), Sg_k \gg_{-1} = 0$$

since $\ll p_1^2 - p_0^2, Sg_k \gg_{-1} = 0$ for all k. On the other hand, by Proposition 5.2, $\|Lg_k\|_{-1}^2 \le (C+1)\|Sg_k\|_{-1}^2$, then $a = 0$. □

Corollary 5.2. *For each $g \in \mathcal{C}_0$, there exists a unique constant $a \in \mathbb{R}$ such that*

$$g + a(p_1^2 - p_0^2) \in \overline{L\mathcal{C}}.$$

By this corollary, it is obvious that there exists a sequence of local functions F_K in $\mathcal{C}$ such that

$$\|W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2) - L^\gamma F_K\|_{-1} \to 0$$

as $K \to \infty$ which conclude the first statement of Lemma 3.3. We prove the rest of the claim of Lemma 3.3 in Proposition 10.2 in Section 10.

6. Boltzmann-Gibbs principle

In this section, we prove Lemma 3.2. First, recall that for each $E > 0$, $\beta(E)$ is the inverse of the function $E(\beta) = \frac{1}{2\beta} + \tilde{V}(\beta)$. Then, by simple calculations, we have

$$\frac{d}{dE}\langle p_1^2 \rangle_{\beta(E)} = \frac{d}{dE}\Big(\frac{1}{\beta(E)}\Big) = \frac{-1}{\beta^2}\left(\frac{dE(\beta)}{d\beta}\right)^{-1}$$
$$= \frac{-1}{\beta(E)^2}\left\{-\frac{1}{2\beta(E)^2} + \tilde{V}'(\beta(E))\right\}^{-1} = \frac{1}{\chi(\beta(E))\beta(E)^2}.$$

Now, we can rewrite the term $I_{N,t}^3(H)$ as

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})[p_i^2 - h(E) - h'(E)(\mathcal{E}_i - E)]\, ds$$

where $h(E) = \frac{1}{\beta(E)} = \langle p_1^2 \rangle_{\beta(E)}$, and $H_N'' = \Delta^N H$. Lemma 3.2 follows from standard arguments (cf. [12]). We sketch it here for completeness.

Here one can introduce a further average in a microscopic block of length $K << N$ and substitute the following expression

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})\tau_i\varphi_K(\omega_s)\, ds$$

with $\varphi_K = \frac{1}{2K+1}\sum_{|j|\le K}[p_j^2 - h(E) - h'(E)(\mathcal{E}_j - E)]$. Using Schwarz inequality, the difference can be estimated to be of order K/N.

Define $\hat\varphi_K = \varphi_K - \langle\varphi_K\rangle_{\Lambda_K,\bar{\mathcal{E}}_K}$, with $\bar{\mathcal{E}}_K = (2K+1)^{-1}\sum_{j\in\Lambda_K}\mathcal{E}_j$. By (4.1)

$$\begin{aligned}\mathbb{E}_{\nu_\beta}&\left(\sup_{0\le t\le T}\left[\int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})\tau_i\hat\varphi_K(\omega_s)\, ds\right]^2\right)\\ &\le 16T\sup_f\left\{\frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})\langle f\tau_i\hat\varphi_K\rangle - N^2\gamma < f,(-S_N)f>\right\}.\end{aligned} \tag{6.1}$$

By the spectral gap on $-S_{\Lambda_K}$, we can define $U_K = (-S_{\Lambda_K})^{-1}\hat\varphi_K$. Then

$$\begin{aligned}&\frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})\langle f\tau_i\hat\varphi_K\rangle\\ &\qquad\le\left(\frac{1}{N}\sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})^2\langle\hat\varphi_K U_K\rangle\right)^{1/2}(N < f,(-S_{\Lambda_K})f>)^{1/2}.\end{aligned}$$

Consequently the right hand side of (6.1) is bounded by

$$\begin{aligned}16T\sup_f\Bigg\{&\left(\frac{1}{N}\sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})^2\langle\hat\varphi_K U_K\rangle\right)^{1/2}(N < f,(-S_{\Lambda_K})f>)^{1/2}\\ &-N^2\gamma < f,(-S_{\Lambda_K})f>\Bigg\}\le\frac{C_K}{N}\end{aligned}$$

and it follows that

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\beta}\left(\sup_{0\le t\le T}\left[\int_0^t \frac{1}{\sqrt{N}} \sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})\tau_i\hat\varphi_K(\omega_s)\, ds\right]^2\right) = 0.$$

We are left to estimate the corresponding term with $\langle\varphi_K\rangle_{\Lambda_K,\bar{\mathcal{E}}_K}$. Denote $\bar\varphi_K = \langle\varphi_K\rangle_{\Lambda_K,\bar{\mathcal{E}}_K}$ and observe that it has support on Λ_k and its variance with respect to ν_β is of order K^{-2}. Then by Schwarz inequality and stationarity

of ν_β

$$
\begin{aligned}
\mathbb{E}_{\nu_\beta}\left(\sup_{0\le t\le T}\left[\int_0^t \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} H_N'(\frac{i}{N})\tau_i\bar\varphi_K(\omega_s)\,ds\right]^2\right)&\\
\le CT^2\langle\left[\frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} H_N''(\frac{i}{N})\tau_i\bar\varphi_K\right]^2\rangle&\\
= \frac{CT^2}{N}\sum_{i,j} H_N''(\frac{i}{N})H_N''(\frac{j}{N})\langle\tau_{i-j}\bar\varphi_K\bar\varphi_K\rangle&\\
\le \frac{CT^2}{2N}\sum_{i,j}(H_N''(\frac{i}{N})^2 + H_N''(\frac{j}{N})^2)\langle\tau_{i-j}\bar\varphi_K\bar\varphi_K\rangle&\\
= \frac{CT^2}{N}\sum_{i} H_N''(\frac{i}{N})^2\sum_j\langle\tau_j\bar\varphi_K\bar\varphi_K\rangle&\\
\le \frac{C'T^2}{N}\sum_{i} H_N''(\frac{i}{N})^2 K\langle\bar\varphi_K^2\rangle&
\end{aligned}
$$

that goes to 0 as $K\to\infty$, uniformly in N.

7. Tightness

The argument exposed above proves the convergence of the finite dimensional distribution of $\mathbb{Q}_N$. In order to conclude the proof of Theorem 1, we need to prove the tightness of the sequence in $C([0,T],\mathfrak{H}_{-k})$. The argument we use is standard. We report it here for completeness.

Compactness follows from the following two statements:

$$
\lim_{A\to\infty}\limsup_{N\to\infty}\mathbb{P}_{\nu_\beta^N}\left(\sup_{0\le t\le T}\|Y_t^N\|_{-k}\ge A\right) = 0, \tag{7.1}
$$

$$
\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}_{\nu_\beta^N}\left(w_{-k}(Y^N,\delta)\ge\epsilon\right) = 0 \qquad \forall\epsilon>0. \tag{7.2}
$$

where $\|\cdot\|_{-k}$ is the norm in $\mathfrak{H}_{-k}$ and $w_{-k}(Y^N,\delta)$ is the corresponding modulus of continuity in $C([0,T],\mathfrak{H}_{-k})$. We recall that $\|\cdot\|_{-k}$ can be written as

$$
\|Y\|_{-k}^2 = \sum_{n\ge 1}(\pi n)^{-2k}|Y(e_n)|^2
$$

with $e_n(x) = \sqrt{2}\sin(\pi n x)$.

Recall the decomposition of Y_t^N given by (3.3):

$$
Y_t^N(e_n) = Y_0^N(e_n) + (\pi n)^2 D\int_0^t Y_s^N(e_n)ds + M^1_{N,F_K,t}(e_n) + Z_{N,F_K,t}(e_n)
$$

where $E_{\nu_\beta^N}\left(\sup_{0\le t\le T}(Z_{N,F_K,t}(e_n))^2\right)$ can be estimated by the proof of Lemmas 3.1, 3.2. On the other hand, $E_{\nu_\beta^N}\left((M^1_{N,F_K,t}(e_n))^2\right)$ can be computed explicitly. Then, for $k>5/2$, (7.1) and (7.2) follows by standard arguments (cf. [12]).

8. Sector condition in $\mathcal{H}_{-1}$

In this section, we show the sector condition in $\mathcal{H}_{-1}$ which is the key point to apply the nongradient method for asymmetric processes.

First, we prepare a useful lemma.

Lemma 8.1. *For all $f, g \in \mathcal{C}$,*

$$\ll Sf, Ag \gg_{-1} = -\langle \Gamma_f (X_0 - Y_{0,1}) \Gamma_g \rangle = \langle \Gamma_g (X_0 - Y_{0,1}) \Gamma_f \rangle.$$

Proof. By the first identity of (5.2),

$$\begin{aligned} - \ll Sf, Ag \gg_{-1} = \ll f, Ag \gg_* &= \sum_{i \in \mathbb{Z}} \langle \tau_i f, Ag \rangle = \sum_{i,j \in \mathbb{Z}} \langle \tau_i f, (X_j - Y_{j,j+1}) g \rangle \\ &= \sum_{i,k \in \mathbb{Z}} \langle \tau_i f, (X_{k+i} - Y_{k+i,k+i+1}) g \rangle = \sum_{i,k \in \mathbb{Z}} \langle \tau_i f, \tau_i ((X_k - Y_{k,k+1}) \tau_{-i} g) \rangle \\ &= \sum_{k \in \mathbb{Z}} \langle f, (X_k - Y_{k,k+1}) \Gamma_g \rangle = \sum_{k \in \mathbb{Z}} \langle \tau_k f, (X_0 - Y_{0,1}) \Gamma_g \rangle = \langle \Gamma_f (X_0 - Y_{0,1}) \Gamma_g \rangle. \end{aligned}$$

□

Define $\tilde{\Gamma}_f$ as $\tilde{\Gamma}_f = \sum_{|i| \le s_f + 1} \tau_i f$. Observe that in the above expression, one can restrict the definition of Γ_f and Γ_g as finite sums, namely, we can replace them by $\tilde{\Gamma}_f$ and $\tilde{\Gamma}_g$. Decompose now $\tilde{\Gamma}_f = \tilde{\Gamma}_f^e + \tilde{\Gamma}_f^o$, where $\tilde{\Gamma}_f^e$ is even in p_0 and $\tilde{\Gamma}_f^o$ is odd. Observe that the vector fields X_0 and $Y_{0,1}$ change the parity of p_0, so we have

$$\langle \tilde{\Gamma}_f (X_0 - Y_{0,1}) \tilde{\Gamma}_g \rangle = \langle \tilde{\Gamma}_f^o (X_0 - Y_{0,1}) \tilde{\Gamma}_g^e \rangle - \langle \tilde{\Gamma}_g^o (X_0 - Y_{0,1}) \tilde{\Gamma}_f^e \rangle.$$

Applying Schwarz inequality, we can bound the above expression by

$$\langle (\tilde{\Gamma}_f^o)^2 \rangle^{1/2} \langle [(X_0 - Y_{0,1}) \tilde{\Gamma}_g^e]^2 \rangle^{1/2} + \langle (\tilde{\Gamma}_g^o)^2 \rangle^{1/2} \langle [(X_0 - Y_{0,1}) \tilde{\Gamma}_f^e]^2 \rangle^{1/2}$$

and applying the spectral gap for X_0^2 plus Schwarz inequality again, the last term is bounded by

$$C \{ \langle (X_0 \tilde{\Gamma}_f^o)^2 \rangle^{1/2} \langle (X_0 \tilde{\Gamma}_g^e)^2 + (Y_{0,1} \tilde{\Gamma}_g^e)^2 \rangle^{1/2} + \langle (X_0 \tilde{\Gamma}_g^o)^2 \rangle^{1/2} \langle (X_0 \tilde{\Gamma}_f^e)^2 + (Y_{0,1} \tilde{\Gamma}_f^e)^2 \rangle^{1/2} \}$$

with some positive constant C.

Recall that

$$\| Sf \|_{-1}^2 = \frac{\gamma}{2} \langle (Y_{0,1} \tilde{\Gamma}_f)^2 + (X_0 \tilde{\Gamma}_f)^2 \rangle = \frac{\gamma}{2} \langle (Y_{0,1} \tilde{\Gamma}_f^e)^2 + (Y_{0,1} \tilde{\Gamma}_f^o)^2 + (X_0 \tilde{\Gamma}_f^e)^2 + (X_0 \tilde{\Gamma}_f^o)^2 \rangle.$$

Then we obtain the sector condition:

Lemma 8.2 (sector condition)**.** *There exists a positive constant C such that for all $f, g \in \mathcal{C}$,*

$$| \ll Sf, Ag \gg_{-1} | \le C \| Sf \|_{-1} \| Sg \|_{-1}.$$

9. Closed forms

In this section, to complete the proof of Lemma 4.3, we introduce the notion of closed forms and give a characterization of them. We generalize some ideas developed in the PhD thesis of Hernandez [11], where microcanonical surfaces were given by spheres, following the general setup of the seminal work of Varadhan [19] (see also [12], appendix 3, section 4). The nonlinearity of our interaction reflected in the nonconstant curvature of our microcanonical manifolds, requires some substantial modification of the original approach. It is in this section that we will make use of the spectral gap estimate proved in section 12.

Let us decompose $\mathcal{C} = \cup_{k\geq 1}\mathcal{C}_k$, where $\mathcal{C}_k$ is the space of functions $F \in \mathcal{C}$ depending only on the variables $(p_i, r_i)_{-k\leq i\leq k}$. Given $F \in \mathcal{C}_k$ recall the definition of the formal sum

$$\Gamma_F(p,r) = \sum_{j=-\infty}^{\infty} \tau_j F(p,r)$$

and that for every $i \in \mathbb{Z}$ the expressions

$$\frac{\partial \Gamma_F}{\partial p_i}(p,r) = \sum_{i-k\leq j\leq i+k} \frac{\partial}{\partial p_i}\tau_j F(p,r)$$

and

$$\frac{\partial \Gamma_F}{\partial r_i}(p,r) = \sum_{i-k\leq j\leq i+k} \frac{\partial}{\partial r_i}\tau_j F(p,r)$$

are well defined. The formal invariance $\Gamma_F(\tau_i(p,r)) = \Gamma_F(p,r)$ leads us to the relation

$$\frac{\partial \Gamma_F}{\partial p_i}(p,r) = \frac{\partial \Gamma_F}{\partial p_0}(\tau_i(p,r)). \tag{9.1}$$

Remember that $Y_{i,j} = p_i\partial_{r_j} - V'(r_j)\partial_{p_i}$ and $X_i := Y_{i,i}$. Given $F \in \mathcal{C}$ and $i \in \mathbb{Z}$, $X_i(\Gamma_F)$ and $Y_{i,i+1}(\Gamma_F)$ are well defined and satisfy

$$X_i(\Gamma_F)(p,r) = \tau_i X_0(\Gamma_F)(p,r), \quad Y_{i,i+1}(\Gamma_F)(p,r) = \tau_i Y_{0,1}(\Gamma_F)(p,r).$$

Now we consider the following linear space

$$\mathcal{B} = \{(X_0(\Gamma_F), Y_{0,1}(\Gamma_F)) \in L^2(\nu_\beta)\times L^2(\nu_\beta) : F \in \mathcal{C}\}.$$

We denote by $\mathfrak{H}$ the linear space generated by the closure of $\mathcal{B}$ in $L^2(\nu_\beta)\times L^2(\nu_\beta)$ and $(0, p_0V'(r_1))$

$$\mathfrak{H} \;=\; \overline{\mathcal{B} + \{(0, p_0 V'(r_1))\}}. \tag{9.2}$$

First, we observe that defining a vector-valued function $\xi = (\xi^0, \xi^1)$ as $(X_0(\Gamma_F), Y_{0,1}(\Gamma_F))$ for $F \in \mathcal{C}$ or $(0, p_0V'(r_1))$, the following properties are satisfied:

i) $X_i(\tau_j\xi^0) = X_j(\tau_i\xi^0)$ for all $i, j \in \mathbb{Z}$,
ii) $Y_{i,i+1}(\tau_j\xi^1) = Y_{j,j+1}(\tau_i\xi^1)$ for all $i, j \in \mathbb{Z}$,
iii) $X_i(\tau_j\xi^1) = Y_{j,j+1}(\tau_i\xi^0)$ if $\{i\}\cap\{j, j+1\} = \varnothing$,
iv) $p_i[X_i(\tau_i\xi^1) - Y_{i,i+1}(\tau_i\xi^0)] = V'(r_{i+1})\tau_i\xi^0 - V'(r_i)\tau_i\xi^1$ for all $i \in \mathbb{Z}$,

v) $V'(r_{i+1})[X_{i+1}(\tau_i\xi^1) - Y_{i,i+1}(\tau_{i+1}\xi^0)] = V''(r_{i+1})p_{i+1}\tau_i\xi^1 - V''(r_{i+1})p_i\tau_{i+1}\xi^0$ for all $i \in \mathbb{Z}$.

We call a weakly closed form (or *germ* of a weakly closed form, cf. [12]), a couple of functions $\xi = (\xi^0, \xi^1) \in L^2(\nu_\beta) \times L^2(\nu_\beta)$, that satisfy i) to v) in a weak sense. A smooth approximation of weakly closed form is not necessarily closed, and some type of Hodge decomposition is needed. This will be done only after localization in the proof of the following theorem that is the main result of this section.

Theorem 3. *If $\xi = (\xi^0, \xi^1) \in L^2(\nu_\beta) \times L^2(\nu_\beta)$ satisfies conditions i) to v) in a weak sense, then $\xi \in \mathfrak{H}$.*

Proof. The goal is to find a sequence $(F_L)_{L\ge 1}$ in $\mathcal{C}$ such that

$$(\xi^0 - X_0(\Gamma_{F_L}), \xi^1 - Y_{0,1}(\Gamma_{F_L})) \underset{L\to\infty}{\longrightarrow} (0, cp_0V'(r_1))$$

in $L^2(\nu_\beta) \times L^2(\nu_\beta)$ for some constant c.

First, observe that for a function $F \in \mathcal{C}_k$ we can rewrite, by using (9.1),

$$X_0(\Gamma_F) = \sum_{i=-k}^{k} X_i(F)(\tau_{-i}(p,r)) \tag{9.3}$$

and

$$\begin{aligned} Y_{0,1}(\Gamma_F) = \sum_{i=-k}^{k-1} Y_{i,i+1}(F)(\tau_{-i}(p,r)) - \left(V'(r_{k+1})\frac{\partial F}{\partial p_k}\right)(\tau_{-k}p) \\ + \left(p_{-k-1}\frac{\partial F}{\partial r_{-k}}\right)(\tau_{k+1}(p,r)). \end{aligned} \tag{9.4}$$

We define for $m = 0, 1$

$$\xi_i^{m,(L)} = \mathbf{E}_{\nu_\beta}[\xi_i^m|\mathcal{F}_L]\varphi\left(\frac{1}{2L+1}\sum_{i=-L}^{L}\{\frac{p_i^2}{2} + V(r_i)\}\right)$$

where $\xi_i^m(p,r) = \tau_i\xi^m(p,r)$, $\mathcal{F}_L$ is the sub σ-field of Ω generated by $(p_i, r_i)_{i=-L}^{L}$ and φ is a smooth positive function with compact support such that $\varphi(E(\beta)) = 1$ and bounded by 1 (we need this cutoff in order to do uniform bounds later). Because ν_β is a product measure and φ satisfies that

$$X_i\varphi\left(\frac{1}{2L+1}\sum_{i=-L}^{L}\{\frac{p_i^2}{2} + V(r_i)\}\right) = 0$$

for $-L \le i \le L$ and

$$Y_{i,i+1}\varphi\left(\frac{1}{2L+1}\sum_{i=-L}^{L}\{\frac{p_i^2}{2} + V(r_i)\}\right) = 0$$

for $-L \le i \le L-1$, the set of functions $\{(\xi_i^{0,L})\}_{-L\le i\le L}$ and $\{(\xi_i^{1,L})\}_{-L\le i\le L-1}$ even satisfies the conditions *i)* to *v)* on the finite set $\{-L, -L+1, \dots, L\}$ if we replace $\tau_i\xi^0$ by $\xi_i^{0,(L)}$ and $\tau_i\xi^1$ by $\xi_i^{1,(L)}$. Therefore, they define a closed form in a weak sense on a finite dimensional space. To obtain a closed form on each microcanonical manifold $\left\{\omega \in (\mathbb{R}^2)^{2L+1}; \sum_{-L}^{L}\mathcal{E}_i = (2L+1)E\right\}$, we first take a

smooth $\mathcal{F}_L$-measurable approximation of $\{(\xi_i^{0,L})\}_{-L\le i\le L}$ and $\{(\xi_i^{1,L})\}_{-L\le i\le L-1}$ in $L^2(\nu_\beta)$, and denote it by $\{(\zeta_i^{0,(L)})\}_{-L\le i\le L}$ and $\{(\zeta_i^{1,(L)})\}_{-L\le i\le L-1}$. For arbitrary chosen $\epsilon_L > 0$, we choose $\{(\zeta_i^{0,L})\}_{-L\le i\le L}$ and $\{(\zeta_i^{1,L})\}_{-L\le i\le L-1}$ satisfying

$$\sum_{i=-L}^{L} \|\xi_i^{0,(L)} - \zeta_i^{0,(L)}\|_{L^2(\nu_\beta)}^2 + \sum_{i=-L}^{L-1} \|\xi_i^{1,(L)} - \zeta_i^{1,(L)}\|_{L^2(\nu_\beta)}^2 \le \frac{\epsilon_L}{4}.$$

Unfortunately, this smooth approximation may not be closed. Therefore, to obtain a smooth closed form, we consider a one-form $\zeta = \sum_{i=-L}^{L} \zeta_i^{0,(L)} dX_i + \sum_{i=-L}^{L-1} \zeta_i^{1,(L)} dY_{i,i+1}$ on each microcanonical manifold. As shown in Section 13, Lie$\{\{X_i, i = -L, \dots, L\}\{Y_{i,i+1}, i = -L, \dots, L-1\}\}$ generates the all tangent space of each microcanonical manifold, so ζ is well-defined on any chart. By the Hodge decomposition (cf. [18]) with respect to a Riemannian structure associated to our microcanonical measure, there exists a smooth function g and a smooth two-form H satisfying

$$\zeta = dg + \delta H$$

since 0 is the only harmonic function on each of these manifolds by the assumptions on V. Let H_i^0 and H_i^1 given by $\delta H = \sum_{i=-L}^{L} H_i^0 dX_i + \sum_{i=-L}^{L-1} H_i^1 dY_{i,i+1}$. Since dg and δH are orthogonal and the set of functions$\{(\xi_i^{0,L})\}_{-L\le i\le L}$ and $\{(\xi_i^{1,L})\}_{-L\le i\le L-1}$ is closed,

$$\begin{aligned}
&\mathbf{E}_{\nu_\beta}\left[\sum_{i=-L}^{L} (H_i^0)^2 + \sum_{i=-L}^{L-1} (H_i^1)^2\right] = \mathbf{E}_{\nu_\beta}\left[\sum_{i=-L}^{L} H_i^0 \zeta_i^{0,(L)} + \sum_{i=-L}^{L-1} H_i^1 \zeta_i^{1,(L)}\right] \\
&= \mathbf{E}_{\nu_\beta}\left[\sum_{i=-L}^{L} H_i^0 (\zeta_i^{0,(L)} - \xi_i^{0,(L)}) + \sum_{i=-L}^{L} H_i^1 (\zeta_i^{1,(L)} - \xi_i^{1,(L)})\right] \\
&\le \mathbf{E}_{\nu_\beta}\left[\sum_{i=-L}^{L} (H_i^0)^2 + \sum_{i=-L}^{L-1} (H_i^1)^2\right]^{1/2} \mathbf{E}_{\nu_\beta}\left[\sum_{i=-L}^{L} (\zeta_i^{0,(L)} - \xi_i^{0,(L)})^2 + \sum_{i=-L}^{L-1} (\zeta_i^{1,(L)} - \xi_i^{1,(L)})^2\right]^{1/2}.
\end{aligned}$$

Therefore, we know that

$$\begin{aligned}
\mathbf{E}_{\nu_\beta}[\sum_{i=-L}^{L} (H_i^0)^2 + \sum_{i=-L}^{L-1} (H_i^1)^2] &= \sum_{i=-L}^{L} \|X_i g - \zeta_i^{0,(L)}\|_{L^2(\nu_\beta)}^2 + \sum_{i=-L}^{L-1} \|Y_{i,i+1} g - \zeta_i^{1,(L)}\|_{L^2(\nu_\beta)}^2 \\
&\le \sum_{i=-L}^{L} \|\xi_i^{0,(L)} - \zeta_i^{0,(L)}\|_{L^2(\nu_\beta)}^2 + \sum_{i=-L}^{L-1} \|\xi_i^{1,(L)} - \zeta_i^{1,(L)}\|_{L^2(\nu_\beta)}^2 \le \frac{\epsilon_L}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{i=-L}^{L} \|X_i g - \xi_i^{0,(L)}\|_{L^2(\nu_\beta)}^2 + \sum_{i=-L}^{L-1} \|Y_{i,i+1} g - \xi_i^{1,(L)}\|_{L^2(\nu_\beta)}^2 \\
&\le 2\Big\{\sum_{i=-L}^{L} \|X_i g - \zeta_i^{0,(L)}\|_{L^2(\nu_\beta)}^2 + \sum_{i=-L}^{L-1} \|Y_{i,i+1} g - \zeta_i^{1,(L)}\|_{L^2(\nu_\beta)}^2\Big\} \\
&+ 2\Big\{\sum_{i=-L}^{L} \|\zeta_i^{0,(L)} - \xi_i^{0,(L)}\|_{L^2(\nu_\beta)}^2 + \sum_{i=-L}^{L-1} \|\zeta_i^{1,(L)} - \xi_i^{1,(L)}\|_{L^2(\nu_\beta)}^2\Big\} \le \epsilon_L.
\end{aligned}$$

From now on, we show that we can choose a $\mathcal{F}_L$-measurable function $g^{(L)}$ which is smooth on each microcanonical manifold (with respect to the vector fields of the tangent space) and satisfies

$$\begin{aligned} X_i(g^{(L)}) &= \xi_i^{0,(L)} + \epsilon_i^{0,(L)} \quad \textit{for} \quad -L \le i \le L, \\ Y_{i,i+1}(g^{(L)}) &= \xi_i^{1,(L)} + \epsilon_i^{1,(L)} \quad \textit{for} \quad -L \le i \le L-1 \end{aligned} \tag{9.5}$$

where $\sum_{i=-L}^{L} \mathbf{E}_{\nu_\beta}[(\epsilon_i^{0,(L)})^2] + \sum_{i=-L}^{L-1} \mathbf{E}_{\nu_\beta}[(\epsilon_i^{1,(L)})^2] \le \epsilon_L$ for arbitrary given $\epsilon_L > 0$. Then, by the spectral gap proved in Section 12, $g^{(L)}$ is in $L^2(\nu_\beta)$, so we have a sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}_L$ such that

$$g_n \to g^{(L)} \ \textit{in} \ L^2(\nu_\beta), \quad X_i(g_n) \to X_i(g^{(L)}) \ \textit{in} \ L^2(\nu_\beta) \quad \textit{for} \quad -L \le i \le L$$

and

$$Y_{i,i+1}(g_n) \to Y_{i,i+1}(g^{(L)}) \ \textit{in} \ L^2(\nu_\beta) \quad \textit{for} \quad -L \le i \le L-1.$$

It means that we can choose a function $g^{(L)}$ in $\mathcal{C}_L$ satisfying (9.5) at the beginning for arbitrary given ϵ_L. From now on, we fix a sequence $\{\epsilon_L\}_L$ that $\epsilon_L \to 0$ as $L \to \infty$. Observe that $g^{(L)} - \mathbf{E}_{\nu_\beta}[g^{(L)} | \mathcal{E}_{-L} + \cdots + \mathcal{E}_L]$ still satisfies (9.5). So we can suppose that $\mathbf{E}_{\nu_\beta}[g^{(L)} | \mathcal{E}_{-L} + \cdots + \mathcal{E}_L = (2L+1)E] = 0$ for every $E > 0$.

Define

$$g^{(L,k)} = \frac{\beta}{2(L+k)\phi_\beta} \mathbf{E}_{\nu_\beta}[p_{-L-k-1}^2 V'(r_{L+k+1})^2 g^{(2L)} | \mathcal{F}_{L+k}]$$

and

$$\widehat{g}^L = \frac{4}{L} \sum_{k=L/2}^{3L/4} g^{(L,k)}$$

where $\phi_\beta := \mathbf{E}_{\nu_\beta}[V'(r_0)^2]$.

Using (9.3) and (9.4) for $g^{(L,k)}$ and then averaging over k we obtain that

$$X_0\left(\sum_{j=-\infty}^{\infty} \tau_j \widehat{g}^L\right) = \xi^0 + \frac{\beta}{\phi_\beta}[I_L^1 + I_L^2 + I_L^3 + I_L^4 + I_L^5]$$

and

$$Y_{0,1}\left(\sum_{j=-\infty}^{\infty} \tau_j \widehat{g}^L\right) = \xi^1 + \frac{\beta}{\phi_\beta}[J_L^1 + J_L^2 + J_L^3 + J_L^4 - R_L^1 + R_L^2],$$

where

$$I_L^1 = \widehat{\sum_{k=L/2}^{3L/4}} \ \widehat{\sum_{i=-L-k}^{L+k-1}} \tau_{-i} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 (\xi_i^{0,(2L)} - \xi_i^{0,(L+k)}) \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}],$$

$$I_L^2 = \widehat{\sum_{k=L/2}^{3L/4}} \ \widehat{\sum_{i=-L-k}^{L+k-1}} \tau_{-i} \{(\xi_i^{0,(L+k)} - \xi_i^0) \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}]\},$$

$$I_L^3 = \widehat{\sum_{k=L/2}^{3L/4}} \ \widehat{\sum_{i=-L-k}^{L+k-1}} \xi^0(p,r) \tau_{-i} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 (\varphi(\mathcal{E}_{-2L,2L}) - 1) | \mathcal{F}_{L+k}],$$

$$I_L^4 = \widehat{\sum_{k=L/2}^{3L/4}} \frac{1}{2(L+K)} \tau_{-L-k} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 \xi_{L+k}^{0,(2L)} \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}],$$

$$I_L^5 = \widehat{\sum_{k=L/2}^{3L/4}} \sum_{i=-L-k}^{L+k} \frac{1}{2(L+K)} \tau_{-i} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 \epsilon_i^{0,(2L)} \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}],$$

$$J_L^1 = \widehat{\sum_{k=L/2}^{3L/4}} \widehat{\sum_{i=-L-k}^{L+k-1}} \tau_{-i} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 (\xi_i^{1,(2L)} - \xi_i^{1,(L+k)}) \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}],$$

$$J_L^2 = \widehat{\sum_{k=L/2}^{3L/4}} \widehat{\sum_{i=-L-k}^{L+k-1}} \tau_{-i} \{ (\xi_i^{1,(L+k)} - \xi_i^1) \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}] \},$$

$$J_L^3 = \widehat{\sum_{k=L/2}^{3L/4}} \widehat{\sum_{i=-L-k}^{L+k-1}} \xi^1(p,r) \tau_{-i} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 (\varphi(\mathcal{E}_{-2L,2L}) - 1) | \mathcal{F}_{L+k}],$$

$$J_L^4 = \widehat{\sum_{k=L/2}^{3L/4}} \widehat{\sum_{i=-L-k}^{L+k-1}} \tau_{-i} \mathbf{E}_{\nu_\beta}[V'(r_{L+k+1})^2 p_{-L-k-1}^2 \epsilon_i^{1,(2L)} \varphi(\mathcal{E}_{-2L,2L}) | \mathcal{F}_{L+k}],$$

$$R_L^1 = \widehat{\sum_{k=L/2}^{3L/4}} \tau_{-L-k} \{ V'(r_{L+k+1}) \frac{\partial}{\partial p_{L+k}} g^{(L,k)} \},$$

$$R_L^2 = \widehat{\sum_{k=L/2}^{3L/4}} \tau_{L+k+1} \{ p_{-L-k-1} \frac{\partial}{\partial r_{-L-k}} g^{(L,k)} \}.$$

Here the hat over the sum symbol means that it is in fact an average, and $\mathcal{E}_{-2L,2L}$ is equal to $\frac{1}{4L+1} \sum_{i=-2L}^{2L} \mathcal{E}_i$.

The proof of the theorem will be concluded in the following way. First we show that the middle terms $I_L^1, I_L^2, I_L^3, I_L^4, I_L^5$ and $J_L^1, J_L^2, J_L^3, J_L^4$ tend to zero in $L^2(\nu_\beta)$. Then, the proof will be concluded by showing the existence of a subsequence of $\{-R_L^1 + R_L^2\}_{L \geq 1}$ weakly convergent to $cp_0 V'(r_1)$ with some constant c.

For the sake of clarity, the proof is divided in three steps. Before that, let us state two remarks.

Remark 9.1. *We know that for $m = 0, 1$, $\boldsymbol{E}_{\nu_\beta}[\xi^m | \mathcal{F}_L] \xrightarrow{L^2} \xi^m$, i.e given $\epsilon > 0$ there exist $L_0 \in \mathbb{N}$ such that*

$$\boldsymbol{E}_{\nu_\beta}[|\xi^m - \xi^{m,(L)}|^2] \leq \epsilon \quad \text{if} \quad L \geq L_0.$$

Moreover, by the translation invariance we have

$$\boldsymbol{E}_{\nu_\beta}[|\xi_i^m - \xi_i^{m,(L)}|^2] \leq \epsilon \quad \text{if} \quad [-L_0 + i, L_0 + i] \subseteq [-L, L].$$

In fact, given $\tau_{-i} A \in \mathcal{F}_L$

$$\int_A \xi_i^{m,(L)} (\tau_{-i}(p,r)) \nu_\beta(dpdr) = \int_{\tau_{-i}(A)} \xi_i^{m,(L)} (p,r) \nu_\beta(dpdr)$$

$$= \int_{\tau_{-i}(A)} \xi_i^m(p,r)\nu_\beta(dpdr) = \int_A \xi_i^m(\tau_{-i}(p,r))\nu_\beta(dpdr) = \int_A \xi^m(p,r)\nu_\beta(dpdr).$$

In addition, since $\xi_i^{m,(L)}(\tau_{-i}) \in \mathcal{F}_{-L-i}^{L-i}$ *we have*

$$\xi_i^{m,(L)}(\tau_{-i}) = \boldsymbol{E}_{\nu_\beta}[\xi^m|\mathcal{F}_{-L-i}^{L-i}]$$

and therefore

$$\boldsymbol{E}_{\nu_\beta}[|\xi_i^m - \xi_i^{m,(L)}|^2] = \boldsymbol{E}_{\nu_\beta}[|\xi^m - \xi_i^{m,(L)}(\tau_{-i})|^2] \le \boldsymbol{E}_{\nu_\beta}[|\xi^m - \xi_0^{m,(L_0)}|^2].$$

Remark 9.2. *Besides a Strong law of large numbers for* $(p_i^2 V'(r_i)^2)_{i\in\mathbb{Z}}$ *we have*

$$\boldsymbol{E}_{\nu_\beta}\left[\left(\frac{1}{L}\sum_{i=1}^{L} p_i^2 V'(r_i)^2 - \frac{\phi_\beta}{\beta}\right)^2\right] \le \frac{C_\beta}{L}$$

for some finite constant C_β*.*

Step 1. The convergence of the middle terms to 0**.** The convergence to zero as L tends to infinity of I_L^1, I_L^2 and I_L^5 in $L^2(\nu_\beta)$ follows from Schwarz inequality, Remark 9.1, the condition of $\{\epsilon_L\}$ and the fact that φ is a bounded function.
Using the symmetry of the measure about exchanges of variables, I_L^3 can be rewritten as

$$\xi^0(p,r)\sum_{k=L/2}^{\widehat{3L/4}}\sum_{i=-L-k}^{\widehat{L+k-1}}\mathbf{E}_{\nu_\beta}[\sum_{j=1}^{\widehat{L-k}} V'(r_{L+k+j})^2 p_{-L-k-j}^2(\varphi(\mathcal{E}_{-2L,2L})-1)|\mathcal{F}_{L+k}](\tau_{-i}(p,r))$$

and then we decompose it as $I_L^6 + \frac{\phi_\beta}{\beta}I_L^7$, where I_L^6 and I_L^7 are respectively

$$\xi^0(p,r)\sum_{k=L/2}^{\widehat{3L/4}}\sum_{i=-L-k}^{\widehat{L+k-1}}$$
$$\mathbf{E}_{\nu_\beta}[\sum_{j=1}^{\widehat{L-k}}\{V'(r_{L+k+j})^2 p_{-L-k-j}^2 - \frac{\phi_\beta}{\beta}\}(\varphi(\mathcal{E}_{-2L,2L}) - 1)|\mathcal{F}_{L+k}](\tau_{-i}(p,r))$$

and

$$\xi^0(p,r)\sum_{k=L/2}^{\widehat{3L/4}}\sum_{i=-L-k}^{\widehat{L+k-1}}\mathbf{E}_{\nu_\beta}[\varphi(\mathcal{E}_{-2L,2L}) - 1|\mathcal{F}_{L+k}](\tau_{-i}(p,r)).$$

For the first term, observe that

$$|I_L^6|^2 \le |\xi^0(p,r)|^2\sum_{k=L/2}^{\widehat{3L/4}}\sum_{i=-L-k}^{\widehat{L+k-1}}\mathbf{E}_{\nu_\beta}\left[\left(\sum_{j=1}^{\widehat{L-k}}\{V'(r_{L+k+j})^2 p_{-L-k-j}^2 - \frac{\phi_\beta}{\beta}\}\right)^2\right],$$

and the expectation inside the last expression is bounded by $\frac{C_\beta}{L-k}$, so

$$||I_L^6||_{L^2(\nu_\beta)}^2 \le \frac{C_\beta}{L}||\xi^0||_{L^2(\nu_\beta)}^2.$$

For the second term, written explicitly the conditional expectation we see that $|I_L^7|^2$ is bounded by

$$|\xi^0(p,r)|^2 \widehat{\sum_{k=L/2}^{3L/4}} \widehat{\sum_{i=-L-k}^{L+k-1}} \int |\varphi(\frac{1}{4L+1}\sum_{|j|>L+k}\mathcal{E}'_j + \frac{1}{4L+1}\sum_{|j|\le L+k}\mathcal{E}_{j+i}) - 1|^2 d\nu_\beta.$$

We rewrite the integral part as

$$\int |\varphi(\frac{1}{4L+1}\sum_{|j|>L+k}(\mathcal{E}'_j - E_\beta) + \frac{1}{4L+1}\sum_{|j|\le L+k}(\mathcal{E}_{j+i} - E_\beta) + E_\beta) - 1|^2 d\nu_\beta.$$

Using the fact that φ is a Lipschitz positive function bounded by 1 such that $\varphi(E_\beta) = 1$, we obtain that $|I_L^7|^2$ is bounded from above by

$$|\xi^0(p,r)|^2 \widehat{\sum_{k=L/2}^{3L/4}} \widehat{\sum_{i=-L-k}^{L+k-1}} 1 \wedge \int |\frac{1}{4L+1}\sum_{|j|>L+k}(\mathcal{E}'_j - E_\beta) + \frac{1}{4L+1}\sum_{|j|\le L+k}(\mathcal{E}_{j+i} - E_\beta)|^2 d\nu_\beta$$

where $a \wedge b$ denote the minimum of $\{a, b\}$. So, taking expectation and using the Strong law of large numbers together with the dominated convergence theorem, the convergence to zero as L tends to infinity of I_L^3 in $L^2(\nu_\beta)$ is proved.

Same arguments can be applied for J_L^1, J_L^2, J_L^3 and J_L^4. For I_L^4, we can bound the L^2-norm of the term from above by $\frac{C_\beta}{L}\|\xi^0\|^2_{L^2(\nu_\beta)}$ for some constant C_β.

Step 2. The uniform bound of the $L^2(\nu_\beta)$ norms of the boundary terms.

Remember that R_L^1 is defined as

$$\widehat{\sum_{k=L/2}^{3L/4}} \frac{1}{2(L+k)} \tau_{-L-k}\{V'(r_{L+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}V'(r_{L+k+1})^2 \frac{\partial}{\partial p_{L+k}} g^{(2L)} | \mathcal{F}_{L+k}]\}$$
$$= -\widehat{\sum_{k=L/2}^{3L/4}} \frac{1}{2(L+k)}$$
$$\tau_{-L-k}\{V'(r_{L+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}V'(r_{L+k+1})Y_{L+k,L+k+1} g^{(2L)} | \mathcal{F}_{L+k}]\}$$
$$+ \widehat{\sum_{k=L/2}^{3L/4}} \frac{1}{2(L+k)}$$
$$\tau_{-L-k}\{p_{L+k}V'(r_{L+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}V'(r_{L+k+1}) \frac{\partial}{\partial r_{L+k+1}} g^{(2L)} | \mathcal{F}_{L+k}]\}.$$

By Schwarz inequality and (9.5), we can see that the $L^2(\nu_\beta)$ norm of the first term in the right hand side of the last equality is bounded by $\frac{C_\beta}{L}\|\xi^1\|_{L^2(\nu_\beta)}$ for some constant C_β. After an integration by parts, the second term can

be written as

$$\widehat{\sum_{k=L/2}^{3L/4}} \frac{1}{2(L+k)}$$
$$\tau_{-L-k}\{p_{L+k}V'(r_{L+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}\big(\beta V'(r_{L+k+1})^2 - V''(r_{L+k+1})\big)g^{(2L)}|\mathcal{F}_{L+k}]\}. \tag{9.6}$$

Using the symmetry of the measure again, the conditional expectation appearing in the last expression can be rewritten as

$$\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}\widehat{\sum_{j=L+k+1}^{2L}}(\beta V'(r_j)^2 - V''(r_j))(g^{(2L)}\circ \pi_r^{j,L+k+1})|\mathcal{F}_{L+k}]\,,$$

where $\pi_r^{j,L+k+1}$ stands for the exchange operator of r_j and r_{L+k+1}. After that, we decompose the last expression as the sum of the following two terms,

$$\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}\widehat{\sum_{j=L+k+1}^{2L}}(\beta V'(r_j)^2 - V''(r_j))g^{(2L)}|\mathcal{F}_{L+k}],$$

and

$$\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}\widehat{\sum_{j=L+k+1}^{2L}}(\beta V'(r_j)^2 - V''(r_j))(g^{(2L)}\circ \pi_r^{j,L+k+1} - g^{(2L)})|\mathcal{F}_{L+k}].$$

The square of the last expressions are respectively bounded from above by

$$C_\beta L^{-1}\mathbf{E}_{\nu_\beta}[(g^{(2L)})^2|\mathcal{F}_{L+k}], \quad C_\beta \mathbf{E}_{\nu_\beta}[\widehat{\sum_{j=L+k+1}^{2L}}(g^{(2L)}\circ \pi_r^{j,L+k+1} - g^{(2L)})^2|\mathcal{F}_{L+k}]$$

for some constant C_β. Using Schwarz inequality we can see that the square of each term of the sum is respectively bounded from above by

$$\frac{C_\beta}{L^3}\mathbf{E}_{\nu_\beta}[(\widehat{\sum_{k=L/2}^{3L/4}} p^2_{L+k})(g^{(2L)})^2], \tag{9.7}$$

and

$$\begin{aligned}
&\frac{C'_\beta}{L^2}\widehat{\sum_{k=L/2}^{3L/4}}\mathbf{E}_{\nu_\beta}\big[p^2_{L+k}\widehat{\sum_{j=L+k+1}^{2L}}(g^{(2L)}\circ \pi_r^{j,L+k+1} - g^{(2L)})^2\big] \\
&\le \frac{C'_\beta}{L^2}\widehat{\sum_{k=L/2}^{3L/4}}\mathbf{E}_{\nu_\beta}\big[p^2_{L+k}\widehat{\sum_{j=L+k+1}^{2L}} 2\{j-(L+k+1)\}\sum_{i=L+k+1}^{j-1}(g^{(2L)}\circ \pi_r^{i,i+1} - g^{(2L)})^2\big] \\
&\le \frac{C'_\beta}{L}\widehat{\sum_{k=L/2}^{3L/4}}\mathbf{E}_{\nu_\beta}\big[p^2_{L+k}\sum_{i=3L/2+1}^{2L}(g^{(2L)}\circ \pi_r^{i,i+1} - g^{(2L)})^2\big]
\end{aligned}$$

for some constants C_β and C'_β. One can now estimate $\widehat{\sum}_{k=L/2}^{3L/4} p^2_{L+k}$ uniformly because of the cutoff. Using the spectral gap estimate (12.1) proved in Section 12, we can bound (9.7) by a constant.

Finally, we state that we can bound the term $\mathbf{E}_{\nu_\beta}\big[(g^{(2L)}\circ\pi_r^{i,i+1}-g^{(2L)})^2\big]$ by the Dirichlet form of $g^{(2L)}$ which concludes the proof.

Proposition 9.1. *There exists some constant C such that for every smooth function $f:\Omega\to\mathbb{R}$,*

$$\boldsymbol{E}_{\nu_\beta}\big[(f\circ\pi_r^{i,i+1}-f)^2\big]\le C\{\boldsymbol{E}_{\nu_\beta}\big[(X_i f)^2\big]+\boldsymbol{E}_{\nu_\beta}\big[(Y_{i,i+1}f)^2\big]\}.$$

Proof. The change of variables and simple computations conclude the proof. □

Step 3. The existence of a weakly convergent subsequence of $\{R_L^1\}_{L\ge1}$. Firstly, observe that the expression (9.6) is equal to

$$p_0V'(r_1)h_L^1(p_0,r_0,\dots,p_{-7L/2},r_{-7L/2})$$

where

$$h_L^1=\sum_{k=L/2}^{\widehat{3L/4}}\frac{1}{2(L+k)}\tau^{-L-k}\mathbf{E}_{\nu_\beta}[p^2_{-L-k-1}(\beta V'(r_{L+k+1})^2-V''(r_{L+k+1}))g^{(2L)}|\mathcal{F}_{L+k}].$$

On the other hand, we had proved in **Step 2** that $\{p_0V'(r_1)h_L^1\}_{L\ge1}$ is bounded in $L^2(\nu_\beta)$, therefore it contains a weakly convergent subsequence $\{p_0V'(r_1)h_{L'}\}_{L'}$. We can conclude in a similar way that $\{h_L^1\}_{L\ge1}$ is bounded in $L^2(\nu_\beta)$, therefore $\{h_{L'}^1\}_{L'}$ contains a weakly convergent subsequence, whose limit will be denoted by h. It is easy to see that

$$\|X_ih_L^1\|_{L^2(\nu_\beta)}\le\frac{C}{L}\|\xi^0\|_{L^2(\nu_\beta)}\qquad\textit{for}\qquad i\in\{0,-1,-2,\cdots\}$$

and

$$\|Y_{i,i+1}h_L^1\|_{L^2(\nu_\beta)}\le\frac{C}{L}\|\xi^1\|_{L^2(\nu_\beta)}\qquad\textit{for}\qquad \{i,i+1\}\subseteq\{0,-1,-2,\cdots\}$$

which implies that $X_ih=0$ for $i\in\{0,-1,-2,\cdots\}$ and $Y_{i,i+1}h=0$ for $\{i,i+1\}\subseteq\{0,-1,-2,\cdots\}$. Since the function h depends only on $\{p_0,r_0,p_{-1},r_{-1},p_{-2},r_{-2}\cdots\}$ one can show that h is a constant function, let's say c. Taking suitable test functions, we can conclude that in fact $\{p_0V'(r_1)h_{L'}^1\}_{L'}$ converges weakly to $cp_0V'(r_1)$. This proves that for every weakly convergent subsequence of $\{R_L^1\}_{L\ge1}$ there exist a constant c such that the limit is $cp_0V'(r_1)$. Exactly the same can be said about $\{R_L^2\}_{L\ge1}$. □

Remark 9.3. *Observe that the roles of the vector fields X_0 and $Y_{0,1}$ are symmetric, in the sense that changing the definition of the energy of the particle i to $\mathcal{E}_i=p_i^2/2+V(r_{i+1})$ their actions in the boundary terms in the above approximation are exchanged. The space of closed forms does not depend on this choice of the definition of the energy $\mathcal{E}_i$, so we also have the equivalent characterization of the closed forms:*

$$\mathfrak{H}_c\;=\;\overline{\mathcal{B}+\{(p_0V'(r_0),0)\}}.\tag{9.8}$$

This imply that, defining by $\xi_F=(X_0\Gamma_F,Y_{0,1}\Gamma_F)$, a closed form ξ can be approximated by $\xi_F+c_0(p_0V'(r_0),0)$ and by $\xi_G+c_1(0,p_0V'(r_1))$, then $c_0=-c_1=c$ and $F-G=-c\frac{p_0^2}{2}$.

10. Diffusion coefficient

In this section, we describe the diffusion coefficient in several variational formulas and prove the second statement of Lemma 3.3. From Corollary 5.2, there exists a unique number $\tilde{D}(\beta)$ such that

$$W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2) \in \overline{L\mathcal{C}} \quad \text{in} \quad \mathcal{H}_{-1}.$$

Our purpose now is to obtain the explicit formula for $\tilde{D}$. To do this, we follow the argument in [15].

Lemma 10.1. *We have*

$$\mathcal{H}_{-1} = \overline{L\mathcal{C}}|_{\mathcal{N}} \oplus \{W_{0,1}\} = \overline{L^*\mathcal{C}}|_{\mathcal{N}} \oplus \{W_{0,1}^*\}$$

where $W_{0,1}^* := W_{0,1}^S - W_{0,1}^A$ *and* $L^* = S - A$.

Proof. We shall prove the first decomposition since the same arguments apply to the second one. Because we have already proved in Lemma 5.4 that $\overline{L\mathcal{C}}|_{\mathcal{N}}$ has a one-dimensional complementary subspace in $\mathcal{H}_{-1}$, it is sufficient to show that $\mathcal{H}_{-1}$ is generated by $\overline{L\mathcal{C}}$ and the current. Let $h \in \mathcal{H}_{-1}$ so that $\ll h, W_{0,1} \gg_{-1} = 0$ and $\ll h, Lg \gg_{-1} = 0$ for all $g \in \mathcal{C}$. By Proposition 5.1, $h = \lim_{k\to\infty}(aW_{0,1}^S + Sh_k)$ in $\mathcal{H}_{-1}$ for some $a \in \mathbb{R}$ and $h_k \in \mathcal{C}$. In particular,

$$\begin{aligned} \|h\|_{-1}^2 &= \lim_{k\to\infty} \ll aW_{0,1}^S + Sh_k, aW_{0,1}^S + Sh_k \gg_{-1} \\ &= \lim_{k\to\infty} \ll aW_{0,1}^S + Sh_k, aW_{0,1} + Lh_k \gg_{-1} \end{aligned}$$

since $\ll aW_{0,1}^S + Sh_k, aW_{0,1}^A + Ah_k \gg_{-1} = 0$ by Lemma 5.3. On the other hand, by assumption $\ll h, aW_{0,1} + Lh_k \gg_{-1} = 0$ for all k. Also, by Proposition 5.2,

$$\sup_k \|aW_{0,1} + Lh_k\|_{-1}^2 \le 2a^2\|W_{0,1}\|_{-1}^2 + 2(C+1)\sup_k \|Sh_k\|_{-1}^2 := C_h$$

is finite. Therefore,

$$\begin{aligned} \|h\|_{-1}^2 &= \lim_{k\to\infty} \ll aW_{0,1}^S + Sh_k, aW_{0,1} + Lh_k \gg_{-1} \\ &= \lim_{k\to\infty} \ll aW_{0,1}^S + Sh_k - h, aW_{0,1} + Lh_k \gg_{-1} \\ &\le \limsup_{k\to\infty} C_h \|aW_{0,1}^S + Sh_k - h\|_{-1} = 0. \end{aligned}$$

This concludes the proof. □

Now, we can define bounded linear operators $T : \mathcal{H}_{-1} \to \mathcal{H}_{-1}$ and $T^* : \mathcal{H}_{-1} \to \mathcal{H}_{-1}$ as

$$\begin{aligned} T(aW_{0,1} + Lf) &:= aW_{0,1}^S + Sf, \\ T^*(aW_{0,1}^* + L^*f) &:= aW_{0,1}^S + Sf \end{aligned}$$

since

$$\|aW_{0,1} + Lf\|_{-1}^2 = \|aW_{0,1}^* + L^*f\|_{-1}^2 = \|aW_{0,1}^S + Sf\|_{-1}^2 + \|aW_{0,1}^A + Af\|_{-1}^2.$$

We can easily show that T^* is the adjoint operator of T and also we have the relations

$$\ll T(p_1^2 - p_0^2), W_{0,1}^* \gg_{-1} = \ll T^*(p_1^2 - p_0^2), W_{0,1} \gg_{-1} = -\frac{1}{\beta^2},$$

and

$$\ll T(p_1^2 - p_0^2), L^* f \gg_{-1} = \ll T^*(p_1^2 - p_0^2), Lf \gg_{-1} = 0$$

for all $f \in \mathcal{C}$. In particular,

$$\mathcal{H}_{-1} = \overline{L^*\mathcal{C}}|_{\mathcal{N}} \oplus \{T(p_1^2 - p_0^2)\}$$

and there exists a unique number $Q(\beta)$ such that

$$W_{0,1}^* + Q(\beta) T(p_1^2 - p_0^2) \in \overline{L^*\mathcal{C}} \quad \text{in} \quad \mathcal{H}_{-1}.$$

It will turn out later that $Q(\beta) = \tilde{D}(\beta)$.

Lemma 10.2.

$$Q(\beta) = \frac{1}{\beta^2 \|T(p_1^2 - p_0^2)\|_{-1}^2} = \beta^2 \inf_{f \in \mathcal{C}} \|W_{0,1}^* - L^* f\|_{-1}^2. \tag{10.1}$$

Proof. First identity follows from the fact that

$$\ll T(p_1^2 - p_0^2), W_{0,1}^* + Q(\beta) T(p_1^2 - p_0^2) \gg_{-1} = -\frac{1}{\beta^2} + Q(\beta) \|T(p_1^2 - p_0^2)\|_{-1}^2 = 0.$$

Second identity is obtained by the expression

$$\inf_{f \in \mathcal{C}} \|W_{0,1}^* + Q(\beta) T(p_1^2 - p_0^2) - L^* f\|_{-1} = 0$$

since

$$\begin{aligned} &\inf_{f \in \mathcal{C}} \|W_{0,1}^* + Q(\beta) T(p_1^2 - p_0^2) - L^* f\|_{-1}^2 \\ &\quad = \inf_{f \in \mathcal{C}} \|W_{0,1}^* - L^* f\|_{-1}^2 - \frac{2Q(\beta)}{\beta^2} + Q(\beta)^2 \|T(p_1^2 - p_0^2)\|_{-1}^2 \\ &\quad = \inf_{f \in \mathcal{C}} \|W_{0,1}^* - L^* f\|_{-1}^2 - \frac{2Q(\beta)}{\beta^2} + \frac{Q(\beta)}{\beta^2}. \end{aligned}$$

□

By a simple computation, we can show that $\ll Tg, g \gg_{-1} = \ll Tg, Tg \gg_{-1}$ for all $g \in \mathcal{H}_{-1}$, and therefore $(p_1^2 - p_0^2) - T(p_1^2 - p_0^2) \in \overline{L^*\mathcal{C}_0}$ since $(p_1^2 - p_0^2) - T(p_1^2 - p_0^2)$ is orthogonal to $T(p_1^2 - p_0^2)$. By the fact, we obtain the variational formula for $\|T(p_1^2 - p_0^2)\|_{-1}^2$:

Lemma 10.3.

$$\|T(p_1^2 - p_0^2)\|_{-1}^2 = \inf_{f \in \mathcal{C}} \|p_1^2 - p_0^2 - L^* f\|_{-1}^2. \tag{10.2}$$

Proof. By the similar argument with the proof of Proposition 10.2, we have

$$\inf_{f\in\mathcal{C}} \|p_1^2 - p_0^2 - T(p_1^2 - p_0^2) - L^* f\|_{-1}^2 = 0$$

and

$$\begin{aligned}&\inf_{f\in\mathcal{C}} \|p_1^2 - p_0^2 - T(p_1^2 - p_0^2) - L^* f\|_{-1}^2 \\ &= \inf_{f\in\mathcal{C}} \|p_1^2 - p_0^2 - L^* f\|_{-1}^2 - \|T(p_1^2 - p_0^2)\|_{-1}^2\end{aligned}$$

which concludes the proof. □

Proposition 10.1.

$$\tilde{D}(\beta) = \beta^2 \inf_{f\in\mathcal{C}} \|W_{0,1}^* - L^* f\|_{-1}^2 = \frac{1}{\beta^2 \inf_{f\in\mathcal{C}} \|p_1^2 - p_0^2 - L^* f\|_{-1}^2}. \qquad (10.3)$$

Proof. By the definition, $W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2) \in \overline{L\mathcal{C}}$ and therefore

$$\ll W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2), T^*(p_1^2 - p_0^2) \gg_{-1} = -\frac{1}{\beta^2} + \tilde{D}(\beta)\|T(p_1^2 - p_0^2)\|_{-1}^2 = 0.$$

Then, $\tilde{D}(\beta) = Q(\beta)$ follows and we obtain two variational formulas from (10.1) and (10.2). □

Proposition 10.2. *For any sequence F_K in $\mathcal{C}$ such that*

$$\lim_{K\to\infty} \|W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2) - LF_K\|_{-1} = 0,$$

we have

$$\lim_{K\to\infty} [\frac{\gamma}{2}\langle(p_0 V'(r_1) - Y_{0,1}\Gamma_{F_K})^2\rangle + \frac{\gamma}{2}\langle(X_0\Gamma_{F_K})^2\rangle] = \frac{\tilde{D}(\beta)}{\beta^2}.$$

Proof. By the assumption,

$$\lim_{K\to\infty} \|T\{W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2) - LF_K\}\|_{-1} = 0$$

and therefore

$$\lim_{K\to\infty} \|W_{0,1}^S - SF_K\|_{-1}^2 = \tilde{D}(\beta)^2\|T(p_1^2 - p_0^2)\|_{-1}^2.$$

Then, since

$$\tilde{D}(\beta) = Q(\beta) = \frac{1}{\beta^2\|T(p_1^2 - p_0^2)\|_{-1}^2}$$

and

$$\|W_{0,1}^S - SF_K\|_{-1}^2 = \frac{\gamma}{2}\langle(p_0 V'(r_1) - Y_{0,1}\Gamma_{F_K})^2\rangle + \frac{\gamma}{2}\langle(X_0\Gamma_{F_K})^2\rangle,$$

we complete the proof. □

11. Detailed estimates of the diffusion coefficient

In this section, we give some detailed estimates of the diffusion coefficient as a function of γ. Note that they are not necessary to prove our main theorem.

First, we rewrite the variational formula for the diffusion coefficient given by the terms of the norm of $\mathcal{H}_{-1}$ in a tractable way.

Observe that $\mathcal{C}$ is divided into two orthogonal spaces $\mathbb{L}_e$ and $\mathbb{L}_o$ where $\mathbb{L}_e$ is the set of even functions in p and $\mathbb{L}_o$ is the set of odd functions in p. More precisely, for $f \in \mathcal{C}$, $f \in \mathbb{L}_e$ if and only if $f(p,r) = f(-p,r)$ and $f \in \mathbb{L}_o$ if and only if $f(p,r) = -f(-p,r)$ where $(-p)_i = -p_i$ for all i.

Consider two subspaces of $\mathcal{H}_{-1}$ defined as $\mathcal{H}_{-1}^e := \overline{S\mathbb{L}_e|_{\mathcal{N}}} \oplus \{W_{0,1}^S\}$ and $\mathcal{H}_{-1}^o := \overline{S\mathbb{L}_o|_{\mathcal{N}}}$.

Lemma 11.1. *We have*

$$\mathcal{H}_{-1} = \mathcal{H}_{-1}^e \oplus \mathcal{H}_{-1}^o$$

and they are orthogonal to each other. Moreover, $W_{0,1}^A \in \mathcal{H}_{-1}^o$, $Af \in \mathcal{H}_{-1}^o$ *if* $f \in \mathbb{L}_e$ *and* $Af \in \mathcal{H}_{-1}^e$ *if* $f \in \mathbb{L}_o$.

Proof. Straightforward. □

Proposition 11.1.

$$\begin{aligned}\tilde{D}(\beta) = \beta^2 \inf_{f\in\mathbb{L}_e}\sup_{g\in\mathbb{L}_o}\Big\{&\gamma\big[\frac{1}{2}\langle(p_0V'(r_1) - Y_{0,1}\Gamma_f)^2\rangle + \frac{1}{2}\langle(X_0\Gamma_f)^2\rangle\big] \\ &+2\langle(W_{0,1}^A - Af)\Gamma_g\rangle - \gamma\big[\frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle + \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle\big]\Big\}.\end{aligned} \tag{11.1}$$

Proof. We can rewrite the first variational formula for $\tilde{D}(\beta)$ in (10.3) as

$$\begin{aligned}&\beta^2\inf_{f\in\mathcal{C}}\{\|W_{0,1}^S - Sf\|_{-1}^2 + \|W_{0,1}^A - Af\|_{-1}^2\} \\ &= \beta^2 \inf_{f_e\in\mathbb{L}_e}\inf_{f_o\in\mathbb{L}_o}\{\|W_{0,1}^S - Sf_e\|_{-1}^2 + \|Sf_o\|_{-1}^2 + \|W_{0,1}^A - Af_e\|_{-1}^2 + \|Af_o\|_{-1}^2\} \\ &= \beta^2 \inf_{f\in\mathbb{L}_e}\{\|W_{0,1}^S - Sf\|_{-1}^2 + \|W_{0,1}^A - Af\|_{-1}^2\} \\ &= \beta^2 \inf_{f\in\mathbb{L}_e}\sup_{g\in\mathbb{L}_o}\{\|W_{0,1}^S - Sf\|_{-1}^2 - 2 \ll W_{0,1}^A - Af, Sg \gg_{-1} - \|Sg\|_{-1}^2\} \\ &= \beta^2 \inf_{f\in\mathbb{L}_e}\sup_{g\in\mathbb{L}_o}\{\gamma[\frac{1}{2}\langle(p_0V'(r_1) - Y_{0,1}\Gamma_f)^2\rangle + \frac{1}{2}\langle(X_0\Gamma_f)^2\rangle] + 2\langle(W_{0,1}^A - Af)\Gamma_g\rangle \\ &\quad - \gamma[\frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle + \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle]\}.\end{aligned}$$

□

Proposition 11.2.

$$\tilde{D}(\beta) \le \frac{\gamma}{4}\langle V''(r_0)\rangle + \frac{3}{4\gamma}.$$

Proof. Take $f = -\frac{p_0^2}{4}$ in the variational formula (11.1), then we have

$$\tilde{D}(\beta) \le \beta^2 \sup_{g\in\mathbb{L}_o}\{\frac{\gamma}{4}\langle p_0^2 V'(r_0)^2\rangle + 2\langle\{W_{0,1}^A + A(\frac{p_0^2}{4})\}\Gamma_g\rangle - \gamma[\frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle + \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle]\}$$
$$= \frac{\gamma}{4}\langle V''(r_0)\rangle + \frac{\beta^2}{\gamma}\sup_{g\in\mathbb{L}_o}\{2\langle(W_{0,1}^A + A(\frac{p_0^2}{4}))\Gamma_g\rangle - \frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle - \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle\}.$$

Since $W_{0,1}^A = Y_{0,1}(\frac{p_0^2}{2})$,

$$\sup_{g\in\mathbb{L}_o}\{2(W_{0,1}^A + A(\frac{p_0^2}{4}))\Gamma_g\rangle - \frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle - \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle\}$$
$$= \sup_{g\in\mathbb{L}_o}\{-\frac{1}{2}\langle p_0^2, X_0\Gamma_g\rangle - \frac{1}{2}\langle p_0^2, Y_{0,1}\Gamma_g\rangle - \frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle - \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle\}$$
$$= \sup_{g\in\mathbb{L}_o}\{-\frac{1}{2}\langle(X_0\Gamma_g + \frac{p_0^2}{2})^2\rangle - \frac{1}{2}\langle(Y_{0,1}\Gamma_g + \frac{p_0^2}{2})^2\rangle + \frac{1}{4}\langle p_0^4\rangle\} \le \frac{1}{4}\langle p_0^4\rangle.$$

□

Proposition 11.3.

$$\tilde{D}(\beta) \ge \frac{\gamma}{4\beta\langle r_0^2\rangle}.$$

Proof. By the variational formula (11.1)

$$\tilde{D}(\beta) \ge \gamma\beta^2 \inf_{f\in\mathbb{L}_e}\{[\frac{1}{2}\langle(p_0V'(r_1) + Y_{0,1}\Gamma_f)^2\rangle + \frac{1}{2}\langle(X_0\Gamma_f)^2\rangle]\}.$$

Since $\frac{1}{\beta^2} = \langle p_0V'(r_1), p_0r_1\rangle$ and $\langle p_0r_0, X_0(\Gamma_f)\rangle - \langle p_0r_1, Y_{0,1}(\Gamma_f)\rangle = \langle V'(r_0)r_0 - V'(r_1)r_1, \Gamma_f\rangle = 0$ for any $f \in \mathbb{L}_e$, we have

$$\frac{1}{\beta^2} = \langle p_0V'(r_1) - Y_{0,1}(\Gamma_f), p_0r_1\rangle + \langle p_0r_0, X_0(\Gamma_f)\rangle$$

for any $f \in \mathbb{L}_0$. Then, by Schwarz inequality,

$$\frac{1}{\beta^4} \le \inf_{f\in\mathbb{L}_0}\langle(p_0V'(r_1) - Y_{0,1}(\Gamma_f))^2 + (X_0(\Gamma_f))^2\rangle\langle(p_0r_1)^2 + (p_0r_0)^2\rangle$$
$$= \frac{2}{\beta}\langle r_0^2\rangle \inf_{f\in\mathbb{L}_0}\langle(p_0V'(r_1) - Y_{0,1}(\Gamma_f))^2 + (X_0(\Gamma_f))^2\rangle.$$

□

Remark 11.1. *For the harmonic case with $V(r) = \frac{r^2}{2}$, we have an explicit fluctuation-dissipation given by*

$$\begin{aligned} W_{0,1}^A + W_{0,1}^S &= -p_0r_1 + \frac{\gamma}{2}(p_0^2 - r_1^2) \\ &= -\nabla\left[\left(\frac{1}{6\gamma} + \frac{\gamma}{4}\right)p_0^2 + \frac{1}{2}r_0r_1\right] + L\left(\frac{1}{6\gamma}(p_0+p_1)r_1 + \frac{r_1^2}{4}\right) \end{aligned} \tag{11.2}$$

i.e. the diffusion coefficient is given by $D(\beta) = \frac{\gamma}{4} + \frac{1}{6\gamma}$ which does not depend on β.

12. Spectral gap

In this section, we prove the spectral gap estimates for the process of finite oscillators without the periodic boundary condition, which is used in the proof of Theorem 3 in Section 9. We use the following notation:

$$E_{\nu_{L,E}}[\ \cdot\ ] := E_{\nu_\beta^L}[\ \cdot\ |\frac{1}{L}\sum_{i=1}^{L}(\frac{p_i^2}{2}+V(r_i))=E].$$

Recall that we assume that $0<\delta_-\le V''(r)\le\delta_+<\infty$. Then it is easy to see that V satisfies

$$0<d_-\le\Big|\frac{\sqrt{2V(r)}}{V'(r)}\Big|\le d_+<\infty$$

for all $r\in\mathbb{R}\smallsetminus\{0\}$ where $d_-=\frac{\sqrt{\delta_-}}{\delta_+}$ and $d_+=\frac{\sqrt{\delta_+}}{\delta_-}$. Under these assumptions, we can operate the change of variables $(p,r)\to(\mathcal{E},\theta)$ as $\sqrt{\mathcal{E}}\cos\theta=\frac{p}{\sqrt{2}}$ and $\sqrt{\mathcal{E}}\sin\theta=sgn(r)\sqrt{V(r)}$, and we obtain that

$$\int_{\mathbb{R}^2}f(p,r)d\nu_\beta^1=\frac{1}{\sqrt{2\pi\beta^{-1}}Z_\beta}\int_0^\infty\int_0^{2\pi}\tilde{f}(\mathcal{E},\theta)e^{-\beta\mathcal{E}}q(\mathcal{E},\theta)d\mathcal{E}d\theta$$

where $q(\mathcal{E},\theta)=|\frac{\sqrt{2V(r(\mathcal{E},\theta))}}{V'(r(\mathcal{E},\theta))}|$, which satisfies $d_-\le q(\mathcal{E},\theta)\le d_+$ for all $\mathcal{E}$ and θ. Here, $\tilde{f}(\mathcal{E},\theta):=f(p(\mathcal{E},\theta),r(\mathcal{E},\theta))$.

Let $h_\beta(x)dx$ be the probability distribution on $\mathbb{R}_+$ of $p^2/2+V(r)$ under $d\nu_\beta^1$, i.e.

$$\int_{\mathbb{R}^2}g(p^2/2+V(r))d\nu_\beta^1=\int_0^\infty g(x)h_\beta(x)dx$$

for any $g:\mathbb{R}_+\to\mathbb{R}$. Then, since $h_\beta(x)=\frac{1}{\sqrt{2\pi\beta^{-1}}Z_\beta}\int_0^{2\pi}e^{-\beta x}q(x,\theta)d\theta$, we obtain

$$\frac{\delta_-}{\delta_+}\beta e^{-\beta x}\le h_\beta(x)\le\frac{\delta_+}{\delta_-}\beta e^{-\beta x}$$

for all $x>0$.

With these notations, we prepare two lemmas before we state the main result of this section.

Lemma 12.1. *There exists a positive constant C such that*

$$E_{\nu_{1,E}}[(f-E_{\nu_{1,E}}[f])^2]\le CE_{\nu_{1,E}}[(X_1f)^2]$$

for every $E>0$, and every smooth function f.

Proof. By simple computations with the change of variable,

$$E_{\nu_{1,E}}[(f-E_{\nu_{1,E}}[f])^2]=\frac{\int_0^{2\pi}(\tilde{f}(E,\theta)-E_{\nu_{1,E}}[f])^2q(E,\theta)d\theta}{\int_0^{2\pi}q(E,\theta)d\theta}$$

and

$$E_{\nu_{1,E}}[(X_1f)^2]=\frac{\int_0^{2\pi}\{q(E,\theta)^{-1}\partial_\theta\tilde{f}(E,\theta)\}^2q(E,\theta)d\theta}{\int_0^{2\pi}q(E,\theta)d\theta}.$$

Therefore, it sufficient to show that there exists a positive constant C such that

$$\int_0^{2\pi} (\tilde{f}(E,\theta) - E_{\nu_{1,E}}[f])^2 q(E,\theta) d\theta \le C \int_0^{2\pi} (\partial_\theta \tilde{f}(E,\theta))^2 q(E,\theta)^{-1} d\theta$$

for every $E > 0$ and every smooth function f. Then, since $d_- \le q(E,\theta) \le d_+$ for all $E > 0$ and θ, and

$$\int_0^{2\pi} (\tilde{f}(E,\theta) - E_{\nu_{1,E}}[f])^2 q(E,\theta) d\theta \le \int_0^{2\pi} \Big(\tilde{f}(E,\theta) - (\int_0^{2\pi} \tilde{f}(E,\theta) d\theta)\Big)^2 q(E,\theta) d\theta$$

holds for every $E > 0$ and every smooth function f, the desired inequality follows from the Poincaré inequality. □

Lemma 12.2. *There exist positive constants $0 < c \le C < \infty$ such that*

$$c\, E \le \alpha_i(E) \le CE$$

for all $E > 0$ and for $i = 1, 2$ where $\alpha_1(E) := E_{\nu_{1,E}}[p_1^2]$ and $\alpha_2(E) := E_{\nu_{1,E}}[V'^2(r_1)]$.

Proof. By the change of variables introduced above,

$$E_{\nu_{1,E}}[p_1^2] = \frac{\int_0^{2\pi} 2E \cos\theta^2 q(E,\theta) d\theta}{\int_0^{2\pi} q(E,\theta) d\theta}$$

and it is easy to show that $\frac{d_-}{d_+} E \le E_{\nu_{1,E}}[p_1^2] \le 2E$. Similarly,

$$E_{\nu_{1,E}}[V'^2(r_1)] \le \frac{2}{d_-^2} E_{\nu_{1,E}}[V(r_1)] = \frac{2\int_0^{2\pi} E \sin\theta^2 q(E,\theta) d\theta}{d_-^2 \int_0^{2\pi} q(E,\theta) d\theta} \le \frac{2E}{d_-^2}$$

and

$$E_{\nu_{1,E}}[V'^2(r_1)] \ge \frac{2}{d_+^2} E_{\nu_{1,E}}[V(r_1)] = \frac{2\int_0^{2\pi} E \sin\theta^2 q(E,\theta) d\theta}{d_+^2 \int_0^{2\pi} q(E,\theta) d\theta} \ge \frac{d_- E}{d_+^3}.$$

□

The following is the main theorem in this section.

Theorem 4. *There exists a positive constant C such that*

$$E_{\nu_{L,E}}[f^2] \le C \sum_{k=1}^{L} E_{\nu_{L,E}}[(X_k f)^2] + CL^2 \sum_{k=1}^{L-1} E_{\nu_{L,E}}[(Y_{k,k+1} f)^2] \tag{12.1}$$

for every positive integer L, every $E > 0$, and every smooth function f satisfying $E_{\nu_{L,E}}[f] = 0$.

Proof. We start the proof by the usual martingale decomposition. Let $\mathcal{G}_k$ be the σ-field generated by variables $\{\mathcal{E}_1, \dots, \mathcal{E}_k, p_{k+1}, r_{k+1}, \dots, p_L, r_L\}$. Define $f_k := E_{\nu_{L,E}}[f|\mathcal{G}_k]$ for $k = 0, 1, \cdots, L$. Note that $f_0 = f$ and $f_L = f_L(\mathcal{E}_1, \dots, \mathcal{E}_L)$. Then, we obtain

$$E_{\nu_{L,E}}[f^2] = \sum_{k=0}^{L-1} E_{\nu_{L,E}}[(f_k - f_{k+1})^2] + E_{\nu_{L,E}}[f_L^2].$$

We analyze each term separately.

By Lemma 12.1, for any k

$$E_{\nu_{L,E}}[(f_k - f_{k+1})^2|\mathcal{G}_k] \le C E_{\nu_{L,E}}[(X_{k+1}f_k)^2|\mathcal{G}_k]$$

and therefore we have

$$\begin{aligned} E_{\nu_{L,E}}[f^2] &\le C\sum_{k=1}^{L} E_{\nu_{L,E}}[(X_k f_{k-1})^2] + E_{\nu_{L,E}}[f_L^2] \\ &\le C\sum_{k=1}^{L} E_{\nu_{L,E}}[(X_k f)^2] + E_{\nu_{L,E}}[f_L^2]. \end{aligned}$$

So we are left to estimate $E_{\nu_{L,E}}[f_L^2]$ in terms of the Dirichlet form $\sum_{k=1}^{L-1} E_{\nu_{L,E}}[(Y_{k,k+1}f_L)^2]$.

Observe that $Y_{k,k+1}f_L = p_k V'(r_{k+1})\left(\partial_{\mathcal{E}_k} - \partial_{\mathcal{E}_{k+1}}\right) f_L(\mathcal{E}_1,\dots,\mathcal{E}_L)$. Since $\nu_{L,E}$ is the conditional probability of the product measure ν_β^L,

$$E_{\nu_{L,E}}[p_k^2 V'(r_{k+1})^2|\mathcal{G}_L] = E_{\nu_{1,\mathcal{E}_k}}[p^2]E_{\nu_{1,\mathcal{E}_{k+1}}}[V'^2(r)] = \alpha_1(\mathcal{E}_k)\alpha_2(\mathcal{E}_{k+1}).$$

By Lemma 12.2, the Dirichlet form $\sum_{k=1}^{L-1} E_{\nu_{L,E}}[(Y_{k,k+1}f_L)^2]$, is equivalent to

$$\sum_{k=1}^{L-1} E_{\nu_{L,E}}\left[\mathcal{E}_k\mathcal{E}_{k+1}\left\{\left(\partial_{\mathcal{E}_k} - \partial_{\mathcal{E}_{k+1}}\right) f_L\right\}^2\right].$$

Now the problem is reduced to the estimates of the spectral gap for the energy dynamics depending only on variables $\mathcal{E}_1,\dots,\mathcal{E}_L$. Since we can write the probability distribution $\nu_{L,E}(\cdot|\mathcal{G}_L)$ on $\{(\mathcal{E}_1,\dots,\mathcal{E}_L) : \sum_i \mathcal{E}_i = LE\}$ as the product measure $\prod_{i=1}^L h_1(x_i)dx_i$ (or $\prod_{i=1}^L h_\beta(x_i)dx_i$ for any β) conditioned on the same surface, Theorem 5 in the next subsection completes the proof. □

12.1. **Spectral gap for the energy dynamics.** Consider the product measure $\prod_{i=1}^L h_1(x_i)dx_i$ on $\mathbb{R}_+^L$ and $d\mu_{L,E}$ the conditional distribution of it on the surface $\Sigma_{L,E} = \{\sum_{i=1}^L x_i = LE\}$. We have the following expression

$$d\mu_{L,E} = \prod_{i=1}^{L} h(x_i) d\lambda_{L,E}(x_1,\dots,x_L)$$

where $d\lambda_{L,E}$ is the uniform measure on the surface $\Sigma_{L,E}$.

Theorem 5. *There exists a positive constant C such that*

$$E_{\mu_{L,E}}[g^2] \le CL^2\sum_{i=1}^{L-1} E_{\mu_{L,E}}\left[x_i x_{i+1}\left(\partial_{x_i} g - \partial_{x_{i+1}} g\right)^2\right]$$

for every positive integer L, every $E > 0$ and every smooth function $g : \Sigma_{L,E} \to \mathbb{R}$ satisfying $E_{\mu_{L,E}}[g] = 0$.

To prove this, we first refer Caputo's result (Example 3.1 in [6]) and recall that $\delta_-/\delta_+ e^{-x} \le h_1(x) \le \delta_+/\delta_- e^{-x}$. Let $E_{i,j}$ and $D_{i,j}$ be operators defined by $E_{i,j}f = E_{\mu_{L,E}}[f|\mathcal{F}_{i,j}]$ and $D_{i,j}f = E_{i,j}f - f$ where $\mathcal{F}_{i,j}$ is the σ-algebra generated by variables $\{x_k\}_{k\neq i,j}$.

Lemma 12.3 (Caputo, [6])**.** *If $\delta_-/\delta_+ > (3/4)^{1/16}$, then there exists a positive constant C such that*

$$E_{\mu_{L,E}}[g^2] \le \frac{C}{L}\sum_{i,j=1}^{L} E_{\mu_{L,E}}[(D_{i,j}g)^2]$$

for every $E > 0$, every positive integer L and every smooth function $g : \Sigma_{L,E} \to \mathbb{R}$ satisfying $E_{\mu_{L,E}}[g] = 0$.

Next, we show that we can take a telescopic sum.

Lemma 12.4. *There exists a positive constant C such that*

$$\frac{1}{L}\sum_{i,j=1}^{L} E_{\mu_{L,E}}[(D_{i,j}g)^2] \le CL^2 \sum_{i=1}^{L-1} E_{\mu_{L,E}}[(D_{i,i+1}g)^2]$$

for every $E > 0$, every positive integer L and every smooth function $g : \Sigma_{L,E} \to \mathbb{R}$.

Proof. First, we rewrite the term $E_{i,j}g$ in an integral form:

$$E_{i,j}g(x) = \frac{1}{\Xi_{x_i+x_j}} \int_0^1 g(R^t_{i,j}x)h((x_i+x_j)t)h((x_i+x_j)(1-t))dt$$

where $\Xi_a = \int_0^1 h(at)h(a(1-t))dt$ and $R^t_{i,j}x \in \mathbb{R}^L_+$ is a configuration defined by

$$(R^t_{i,j}x)_k = \begin{cases} x_k & \text{if } k \ne i, j, \\ (x_i+x_j)t & \text{if } k = i, \\ (x_i+x_j)(1-t) & \text{if } k = j. \end{cases}$$

Then, by Schwarz's inequality we have

$$\begin{aligned}(D_{i,j}g(x))^2 &= (\frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(R^t_{i,j}x) - g(x)\}h((x_i+x_j)t)h((x_i+x_j)(1-t))dt)^2 \\ &\le \frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(R^t_{i,j}x) - g(x)\}^2 h((x_i+x_j)t)h((x_i+x_j)(1-t))dt.\end{aligned}$$

Now, we introduce operators $\pi^{i,j}$, $\sigma^{i,j}$ and $\tilde\sigma^{i,j} : \mathbb{R}^L_+ \to \mathbb{R}^L_+$ for $i < j$ as

$$(\pi^{i,j}x)_k = \begin{cases} x_k & \text{if } k \ne i, j, \\ x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \end{cases}$$

$\sigma^{i,j} := \pi^{j-1,j} \circ \pi^{j-2,j-1} \cdots \circ \pi^{i,i+1}$ and $\tilde\sigma^{i,j} := \pi^{i,i+1} \circ \pi^{i+1,i+2} \cdots \circ \pi^{j-1,j}$. With these notations, for any $i < j$, we can rewrite the term $g(R^t_{i,j}x) - g(x)$ as

$$\begin{aligned} g(R^t_{i,j}x) - g(x) &= \{g(\tilde\sigma^{i,j-1}(R^t_{j-1,j}(\sigma^{i,j-1}x))) - g(R^t_{j-1,j}(\sigma^{i,j-1}x))\} \\ &\quad + \{g(R^t_{j-1,j}(\sigma^{i,j-1}x)) - g(\sigma^{i,j-1}x)\} + \{g(\sigma^{i,j-1}x) - g(x)\}.\end{aligned}$$

Therefore, we can bound the term $E_{\mu_{L,E}}[(D_{i,j}g(x))^2]$ from above by

$$\begin{aligned}
&3E_{\mu_{L,E}}[\frac{1}{\Xi_{x_i+x_j}}\int_0^1\{g(\tilde{\sigma}^{i,j-1}(R^t_{j-1,j}(\sigma^{i,j-1}x)))-g(R^t_{j-1,j}(\sigma^{i,j-1}x))\}^2\\
&\qquad h((x_i+x_j)t)h((x_i+x_j)(1-t))dt]\\
&+3E_{\mu_{L,E}}[\frac{1}{\Xi_{x_i+x_j}}\int_0^1\{g(R^t_{j-1,j}(\sigma^{i,j-1}x))-g(\sigma^{i,j-1}x)\}^2\\
&\qquad h((x_i+x_j)t)h((x_i+x_j)(1-t))dt]\\
&+3E_{\mu_{L,E}}[\frac{1}{\Xi_{x_i+x_j}}\int_0^1\{g(\sigma^{i,j-1}x)-g(x)\}^2h((x_i+x_j)t)h((x_i+x_j)(1-t))dt].
\end{aligned}\tag{12.2}$$

We estimate three terms separately. The last term of equation (12.2) is equal to

$$3E_{\mu_{L,E}}[\{g(\sigma^{i,j-1}x)-g(x)\}^2]$$

and therefore bounded from above by

$$3L\sum_{k=i}^{j-2}E_{\mu_{L,E}}[\{g(\pi^{k,k+1}x)-g(x)\}^2].$$

By simple computations, we obtain that

$$\begin{aligned}
&E_{\mu_{L,E}}[\{g(\pi^{k,k+1}x)-g(x)\}^2]\\
&=E_{\mu_{L,E}}[\{g(\pi^{k,k+1}x)-(E_{k.k+1}g)(\pi^{k,k+1}x)+(E_{k.k+1}g)(x)-g(x)\}^2]\\
&\le 2E_{\mu_{L,E}}[\{g(\pi^{k,k+1}x)-(E_{k.k+1}g)(\pi^{k,k+1}x)\}^2]+2E_{\mu_{L,E}}[\{(E_{k.k+1}g)(x)-g(x)\}^2]\\
&=4E_{\mu_{L,E}}[(D_{k.k+1}g)^2].
\end{aligned}$$

By the change of variable with $y=\sigma^{i,j-1}x$, we can rewrite the second term of equation (12.2) as

$$\begin{aligned}
&3E_{\mu_{L,E}}[\frac{1}{\Xi_{y_{j-1}+y_j}}\int_0^1\{g(R^t_{j-1,j}y)-g(y)\}^2h((y_{j-1}+y_j)t)h((y_{j-1}+y_j)(1-t))dt]\\
&\qquad=3E_{\mu_{L,E}}[E_{j,j+1}(g^2)-2gE_{j.j+1}g+g^2]\\
&\qquad=6E_{\mu_{L,E}}[g^2-(E_{j.j+1}g)^2]=6E_{\mu_{L,E}}[(D_{j,j+1}g)^2].
\end{aligned}$$

Similarly, the first term of equation (12.2) is rewritten as

$$\begin{aligned}
&3E_{\mu_{L,E}}[\frac{1}{\Xi_{y_{j-1}+y_j}}\int_0^1\{g(\tilde{\sigma}^{i,j-1}(R^t_{j-1,j}y))-g(R^t_{j-1,j}y)\}^2\\
&\qquad h((y_{j-1}+y_j)t)h((y_{j-1}+y_j)(1-t))dt]\\
&=3E_{\mu_{L,E}}[E_{j,j+1}(\{g\circ\tilde{\sigma}^{i,j-1}-g\}^2)]=3E_{\mu_{L,E}}[\{g\circ\tilde{\sigma}^{i,j-1}-g\}^2].
\end{aligned}$$

In the same way as the first term of (12.2), it is bounded from above by $12L\sum_{k=i}^{j-2}E_{\mu_{L,E}}[(D_{k,k+1}g)^2]$. Therefore, we complete the proof. □

Lemma 12.5. *There exists a constant C such that*

$$E_{\mu_{2,E}}[(D_{1,2}g)^2] \le C E_{\mu_{2,E}}[x_1 x_2(\partial_{x_1} g - \partial_{x_2} g)^2]. \tag{12.3}$$

for every $E > 0$ and every smooth function $g : \Sigma_{2,E} \to \mathbb{R}$.

Proof. Since the both sides of (12.3) do not change if we replace g with $g + a$ for any constant a, it is sufficient to show that the inequality holds for every smooth function $g : \Sigma_{2,E} \to \mathbb{R}$ satisfying $E_{\lambda_{2,E}}[g] = 0$. In particular, since $E_{\mu_{2,E}}[(D_{1,2}g)^2] \le E_{\mu_{2,E}}[g^2]$, it is sufficient to show that

$$E_{\mu_{2,E}}[g^2] \le C E_{\mu_{2,E}}[x_1 x_2(\partial_{x_1} g - \partial_{x_2} g)^2].$$

Note that for any positive function $f : \Sigma_{2,E} \to \mathbb{R}_+$ and for any $E > 0$,

$$(\frac{\delta_-}{\delta_+})^4 E_{\mu_{2,E}}[f] \le E_{\lambda_{2,E}}[f] \le (\frac{\delta_+}{\delta_-})^4 E_{\mu_{2,E}}[f].$$

In fact,

$$E_{\mu_{2,E}}[f] = \frac{1}{\Xi_E} \int_0^1 f(tE, (1-t)E) h(tE) h((1-t)E) dt$$

where $\Xi_E = \int_0^1 h(tE)h((1-t)E)dt$. Then, since $\frac{\delta_-}{\delta_+} \le h(x) \le \frac{\delta_+}{\delta_-}$, the above estimate holds. Now, all we have to show is that, there exists a constant C such that

$$E_{\lambda_{2,E}}[g^2] \le C E_{\lambda_{2,E}}[x_1 x_2(\partial_{x_1} g - \partial_{x_2} g)^2] \tag{12.4}$$

for every $E > 0$ and every smooth function $g : \Sigma_{2,E} \to \mathbb{R}$ satisfying $E_{\lambda_{2,E}}[g] = 0$. By the definition of $\lambda_{2,E}$, the inequality (12.4) is rewritten as

$$\int_0^E g(t)^2 dt \le C \int_0^E t(E-t) g'(t)^2 dt$$

and by a suitable change of variable, the problem is reduced to the case with $E = 1$. Applying Schwarz inequality and changing the order of integration repeatedly, we have

$$\int_0^1 g(t)^2 dt = \int_0^1 \int_0^t \{\int_s^t g'(r)dr\}^2 ds dt \le \int_0^1 \int_0^t (t-s) \int_s^t g'(r)^2 dr ds dt$$
$$= \int_0^1 \int_0^t (tr - \frac{r^2}{2}) g'(r)^2 dr dt = \frac{1}{2} \int_0^1 g'(r)^2 r(1-r) dr.$$

□

Lemma 12.6. *There exists a positive constant C such that*

$$E_{\mu_{L,E}}[(D_{i,i+1}g)^2] \le C E_{\mu_{L,E}}[x_i x_{i+1}(\partial_{x_i} g - \partial_{x_{i+1}} g)^2]$$

for every positive integer L, $i = 1, \dots, L-1$, and every smooth function $g : \Sigma_{L,E} \to \mathbb{R}$.

Proof. By Lemma 12.5,

$$E_{\mu_{L,E}}[(D_{i,i+1}g)^2 | \mathcal{F}_{i,i+1}] \le C E_{\mu_{L,E}}[x_i x_{i+1}(\partial_{x_i} g - \partial_{x_{i+1}} g)^2 | \mathcal{F}_{i,i+1}]$$

holds. Then, by taking the expectation, we complete the proof. □

13. Lie Algebra

We prove here that Lie$\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ generates the all tangent space of $\Sigma_{N,E}$. We have used this property in section 9, for the characterization of the finite dimensional closed forms.

We have

$$[X_{j+1}, Y_{j,j+1}] = V''(r_{j+1}) Z_{j,j+1}$$

where

$$Z_{i,j} = p_i \partial_{p_j} - p_j \partial_{p_i}$$

Since $[Z_{j,j+1}, Z_{j+1,j+2}] = Z_{j,j+2}$ and $V''(r) > \delta > 0$, we have that $Z_{i,j} \in$ Lie$\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ for any i and j.

On the other hand $[Z_{j,i}, X_j] = Y_{i,j}$, and we have enough vector fields to generate the all tangent space.

Remark 13.1. *By the above argument, it is obvious that Lie*$\{\{X_i, i = 1, \dots, N\}\{Y_{i,i+1}, i = 1, \dots, N-1\}\}$ *also generates the all tangent space of* $\Sigma_{N,E}$.

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